GRAPHICAL MODELS IN ALGEBRAIC COMBINATORICS: VERTEX MODELS

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Abstract. These are lecture notes for part of the SLMath summer school "Graphical Models in Algebraic Combinatorics" which tool place from June 23-July 3, 2025. The aim of these lectures is to introduce Yang-Baxter integrable vertex models and give examples of how these vertex models are used in algebraic combinatorics. Topics include the six vertex model, domain wall boundary conditions, five vertex models and their relationship with Schur polynomials, colored vertex models and Littlewood-Richardson coefficients, and finally an overview on limit shapes and asymptotic results. Each lecture is followed by several exercises.

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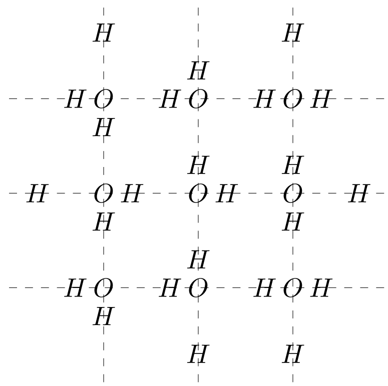
Lecture B2: The Six Vertex Model

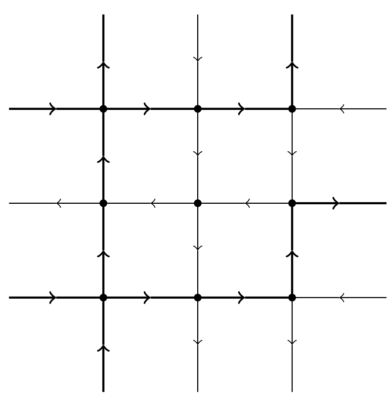
Some Historical Context:
Introduced by Linus Pauling (1930s) for studying 2d ice (specifically at absolute zero).



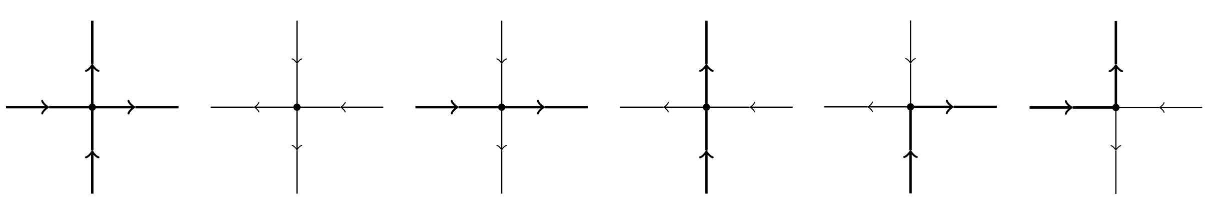
Note:

- Oxygen at vertices of $\mathbb{Z}^2$, hydrogen at each edge. ("Square Ice")
- Each oxygen is bound to two of the hydrogen which are drawn closer to it.

Replace each edge with an arrow pointing towards the vertex the hydrogen belongs to, we get

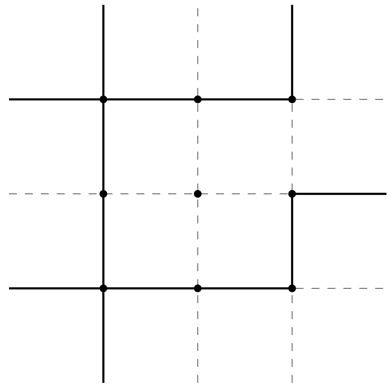


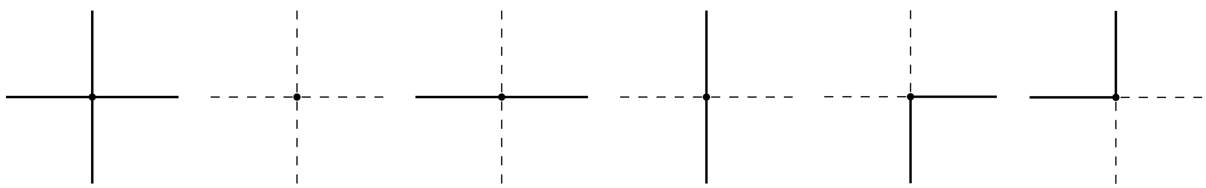
At each vertex there must be two vertices pointing inward and two outward giving $\binom{4}{2} = 6$ possible local configuration at a vertex. Thus the name.



Simple definition: A 6V configuration is an assignment of arrows to the edges of the square lattice (really, any 4-regular graph) such that locally at each vertex we have one of the six possibilities drawn above.

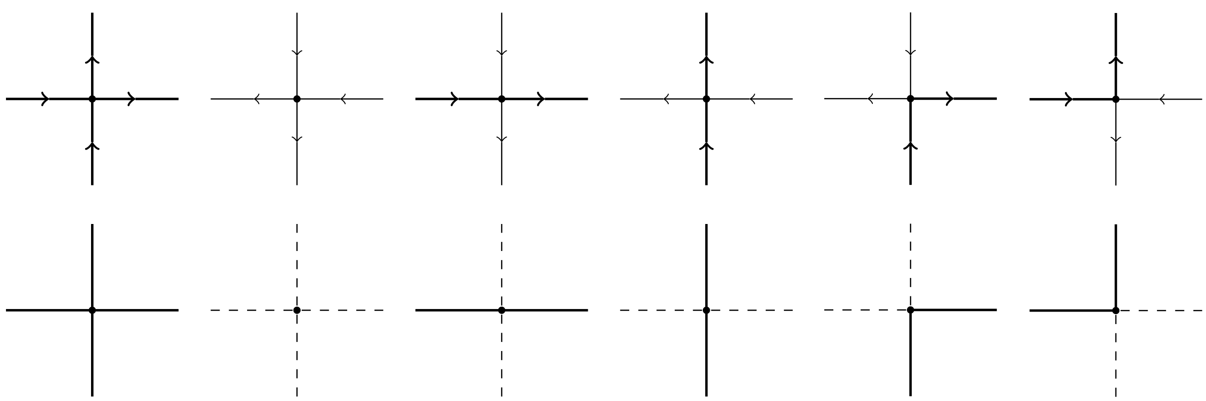
Aside: If we assign each edge either 'path' or 'no path' according to the rule, say, up and right arrows are paths, we can draw the configuration as a collection of up-right paths that can meet at a corner but cannot cross.





Generally, what goes into a vertex model:

- Allowed vertices, i.e. the six vertices of the six vertex model.
- Vertex weights, uniform in the case of square ice, but in general some parameters we assign. The usual parameterization for the six vertex model is given by

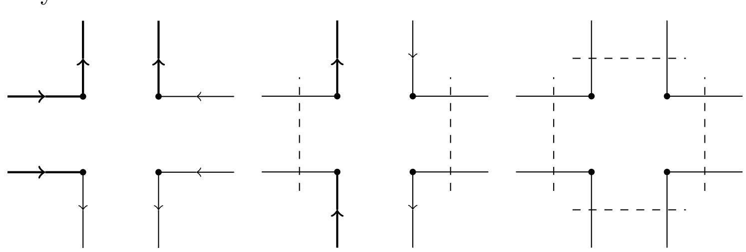


a a b b c c

- A subset D of $\mathbb{Z}^2$ with boundary conditions (assignment of state to each edge that is only adjacent to a single vertex).

For domains, we will usually consider $N \times N$ squares (vertices at (i, j), $i, j \in \{1, 2, \ldots, N\}$ and all adjacent edges).

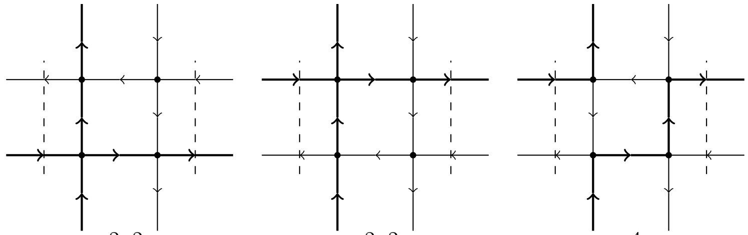
The common boundary conditions are:



fixed cylindrical toroidal

where the edges crossing dashed lines are identified.

For example, in the cylindrical case the possible configurations satisfying the boundary conditions, along with their weights, are



a^2b^2 $\quad$ a^2b^2 $\quad$ c^4

Some definitions:

- Call our domain D. The set of all configurations on D (with some specified boundary condition) will be $\mathcal{C}(D)$.
- Given a configuration $C \in \mathcal{C}(D)$, for each vertex $v \in C$, let
$$w(v) = \text{ weight of a vertex } v$$
where v is one of our six possibilities.
- Given a configuration $C \in \mathcal{C}(D)$, the weight of C is given by
$$w(\mathcal{C}) = \prod_{v \in C} w(v)$$
- We define the *partition function* to be
$$Z_D = \sum_{C \in \mathcal{C}(D)} w(C) = \sum_{C \in \mathcal{C}(D)} \prod_{v \in C} w(v).$$

For the example of the 2×2 cylinder, above, the partition function is
$$Z(a, b, c) = 2a^2b^2 + c^4.$$
Note for uniform weights the partition function just counts the number of possible configurations that satisfy the given boundary conditions.

Pauling was interested in the 'residual entropy', that is, the number of configurations the water molecules could take if you cooled them to absolute zero. Naively, might thing there would be only one possible configuration but he showed that this was not the case. More precisely: Fix periodic b.c. on a $N \times N$ square, assign uniform weights, let Z_N be the partition function. One expects
$$Z_N \sim e^{N^2 c}$$
for some constant c. Lieb showed (1960s) that
$$c = \frac{3}{2} \log \frac{4}{3} \approx \log 1.5396 \approx 0.4315.$$
The key technique was to use the integrability of the model.

R-matrices and row operators:

Before we get to integrability, we need some more definitions/notation. Consider a formal two-dimensional vector space V over $\mathbb{C}$. We can assign each basis vector an arrow, say e_+ is either up or right ((paths) and e_- is down or left (no path). Then V^2 has basis vectors $++$, $+-$, $-+$, and $--$. We can store the weights of the six vertex model in a 4×4 matrix acting from V^2 to V^2 given by

$$R = \begin{array}{c} \\ ++ \\ +- \\ -+ \\ -- \end{array} \begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \end{array}$$

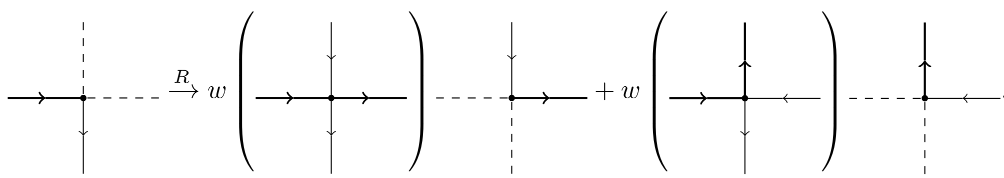
where we view the matrix as taking in the arrows from below and left of the vertex and spitting out a weighted sum of the arrows above and right of the vertex. This is called an R-matrix. The R-matrix describes how to propagate arrows/paths through a vertex. For example, we can write

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

or

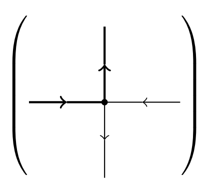
$$Re_+ \otimes e_- = be_+ \otimes e_- + ce_- \otimes e_+$$

or

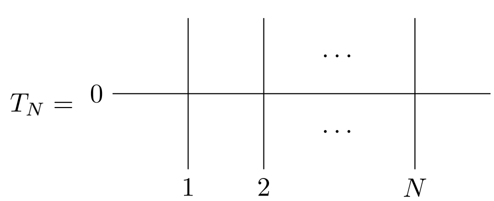


all meaning the same thing.

We can pick out a single matrix entry by multiplying on the right and left by basis (co)vectors. For example, we have

$$(e_+ \otimes e_-)^T Re_+ \otimes e_- = w$$

$$= c.$$

Now consider a row of N vertices:



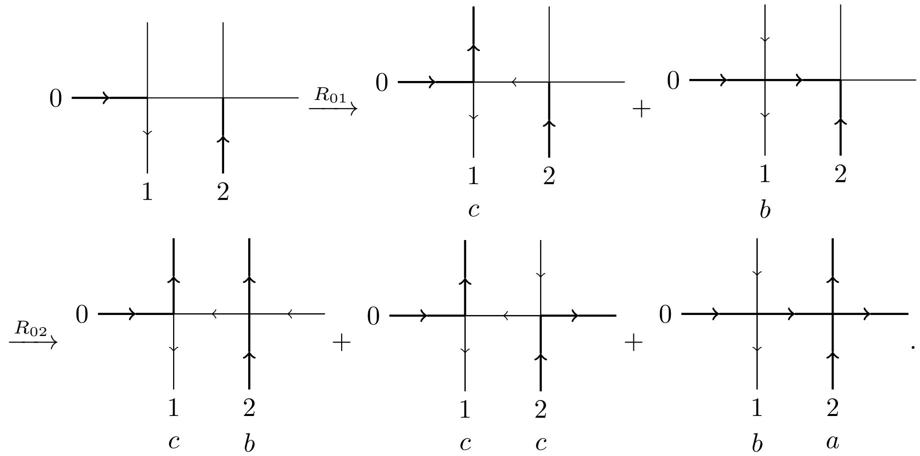
Here T_N is an $2^{N+1} \times 2^{N+1}$ matrix acting from $V_0 \otimes V_1 \otimes \cdots \otimes V_N$ to itself where each V_i is a copy of V and the subscript is used to distinguish which strand the vector space belongs to.

Each matrix entry in this matrix is the partition function of the row with specific boundary condition corresponding to the matrix entry we pick. We may write T_N as a product of R-matrices

$$T_N = R_{0N} \dots R_{02} R_{01}$$

where R_{01} propagates the arrows/paths through the first vertex, then R_{02} propagates them through the second vertex, and so on through the N^{th} vertex.

Let us do an example with $N = 2$ by computing $T_2 e_+ \otimes e_- \otimes e_+$.

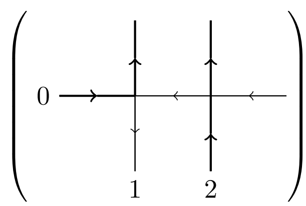


We see that

$$T_2 e_+ \otimes e_- \otimes e_+ = cb e_- \otimes e_+ \otimes e_+ + c^2 e_+ \otimes e_+ \otimes e_- + ab e_+ \otimes e_- \otimes e_+.$$

Picking out a matrix we have , say,

$$(e_- \otimes e_+ \otimes e_+)^T T_2 e_+ \otimes e_- \otimes e_+ = w$$



$$= cb.$$

Integrability:
We are interested in *Yang-Baxter integrable* vertex models. We'll consider three choices of six vertex weight:

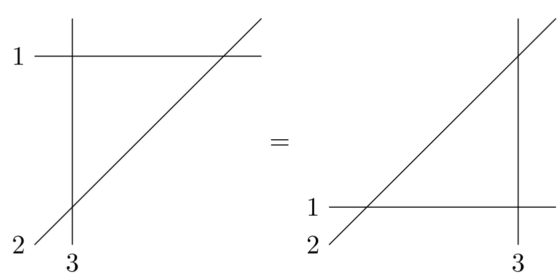
$$w \text{ corresponding to weights } a, b, c$$
$$w' \text{ corresponding to weights } a', b', c'$$
$$w'' \text{ corresponding to weights } a'', b'', c''$$

with corresponding R-matrices R, R', and R''. The Yang-Baxter equation (YBE) is a matrix equation from $V_1 \otimes V_2 \otimes V_3$ to itself given by

$$R_{12} R'_{13} R''_{23} = R''_{23} R'_{13} R_{12}$$

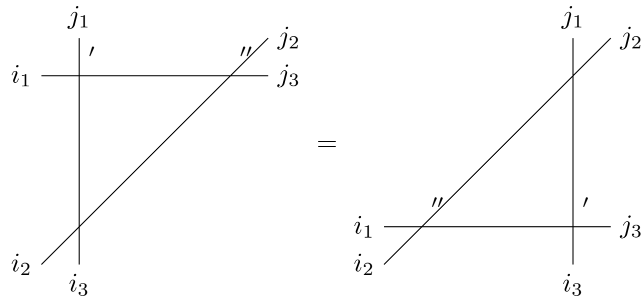
where the primes indicate our three choice of weight and the subscript indicates which copy of our vector space the corresponding R matrix acts on. That is, R_{12} acts as R on $V_1 \otimes V_2$ and as the identity on V_3.

Graphically, this is saying that we have an equality of the two partition functions



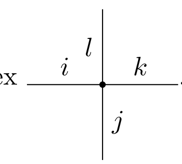
Note that it is common to just draw the picture to mean the corresponding matrix equality.

Each entry in the matrix corresponds to a choice of boundary conditions for the diagrams. We can choose a particular matrix entry (giving us a one equation in the weights that must be satisfied for the YBE to hold) by choosing boundary conditions. So an equivalent way to express the YBE is to say, we must have equality of the partition functions



for every choice of $i_1, i_2, i_3, j_1, j_2, j_3 \in \{+, -\}$ where $+$ corresponds to an up or right arrow (path), and $-$ corresponds to down or left (no path).

We can also write down what the individual equations look like in general. For $i, j, k, l \in \{+, -\}$ define

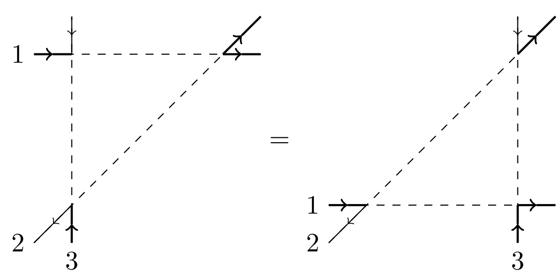
$w(i, j; k, l) =$ weight of the vertex .

(If it doesn't correspond to a valid vertex, we set it equal to zero.) Then the YBE can be written as

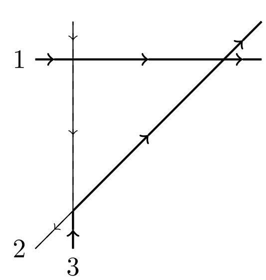
$$\sum_{k_1,k_2,k_3 \in \{+,-\}} w''(i_1, k_1; k_2, j_2) w'(k_2, k_3; j_3, j_2) w(i_2, i_3; k_3, k_1)$$
$$= \sum_{k_1,k_2,k_3 \in \{+,-\}} w''(i_1, i_2; k_3, k_2) w'(k_3, i_3; j_3, k_2) w(k_1, k_2, j_2, j_1)$$

for every choice of $i_1, i_2, i_3, j_1, j_2, j_3 \in \{+, -\}$. This is sometimes called the coordinate form of the YBE. We will stick to writing things in terms of R-matrices or graphically.

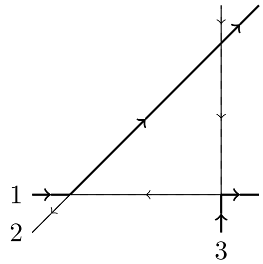
As an example, let's fix boundary conditions $i_1 = i_3 = +$, $i_2 = -$, $j_1 = -$, $j_2 = j_3 = +$. Graphically we have



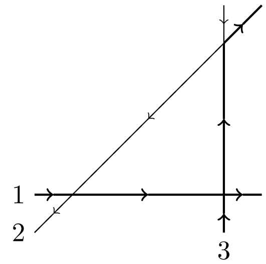
The LHS has only one valid configuration



with weight $ab'c''$. The RHS has two possible configurations



and



so its partition function is

$$cc'b'' + ba'c''.$$

For the YBE to be satisfied we must have that

$$ab'c'' = cc'b'' + ba'c''.$$

This is just one equations out of the $2^3 \times 2^3 = 64$ equations that need to hold for the YBE to be satisfied.

Note: 9 parameters to choose for the weights, 64 equations, a miracle that there are any solutions at all.

Theorem 0.1. *Let $\Delta = \frac{a^2+b^2-c^2}{2ab}$. Given w and w', there exists weights w'' such that the triple of weights satisfy the YBE only if*

$$\Delta = \Delta'.$$

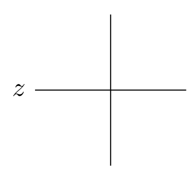
Moreover, we will also have $\Delta = \Delta''$.

There is a particularly nice way to parametrize the six vertex weights given model by

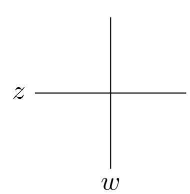
$$\begin{aligned} a(z) &= zq - (zq)^{-1} \\ b(z) &= z - z^{-1} \\ c(z) &= q - q^{-1} \end{aligned}$$

and let $R(z, q)$ be the corresponding R-matrix. Note $\Delta = \frac{q+q^{-1}}{2}$, so weights with different choices of z (called the spectral parameter) satisfy the YBE as long as they have the same value for q. The parametrization given here is due to Baxter.

When drawing vertices we will often label the edges by the spectral parameter. For example

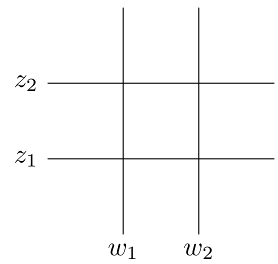


corresponds to a vertex with spectral parameter z. More, generally we can have a row and a column parameters



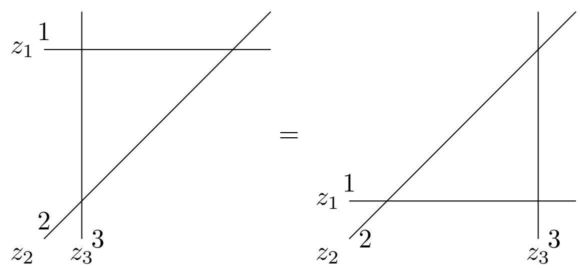
which corresponds to weights with spectral parameter $\frac{z}{w}$.

When drawing domains, the row and column parameters continue along their corresponding strands. For example,



means we use spectral parameter $\frac{z_2}{w_2}$ for the top-right vertex. We will always assume all vertices use the same parameter q.

Let's return to the YBE. We'll draw it slightly differently:



Theorem 0.2. *With this parametrization the YBE*

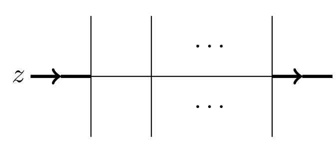
$$R_{12}(z_1/z_2)R_{13}(z_1/z_3)R_{23}(z_2/z_3) = R_{23}(z_2/z_3)R_{13}(z_1/z_3)R_{12}(z_1/z_2)$$

holds.

- We can read off the correct spectral parameters by just following the strands.
- We can think of this as saying that we can swap z_1 and z_2 at the cost of moving the vertex $R_{12}(z_1/z_2)$ from one side to the other.
- Note with this parametrization $\Delta = \frac{1}{2}(q + q^{-1})$.

The proofs of these theorems are left to the first problem set.

Application:
Consider a row operator

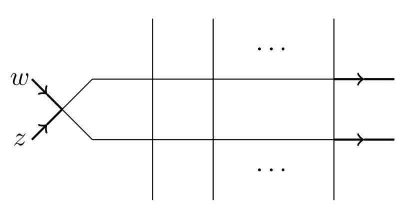
$A(z) =$ 

with N vertices. This can be thought of as a $2^N \times 2^N$ matrix taking $V^{\otimes N}$ to itself. I claim that

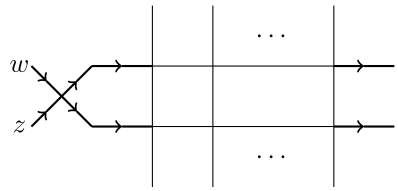
$$A(z)A(w) = A(w)A(z).$$

Note that the product corresponds to a two row vertex model in which each row has a different spectral parameter.

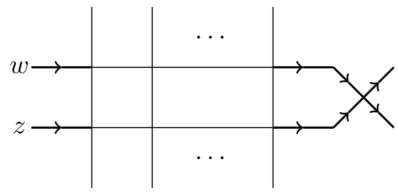
To see the rows commute, consider adding an extra "cross" vertex to the left of the row:



There's only one way the cross can get filled in



which results in $a(w/z)A(z)A(w)$. Now use YBE equation to push the cross from the LHS to the RHS and get



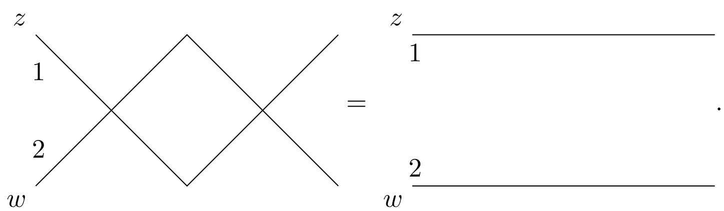
where again we use that there is a unique way to fill in the cross. This give $a(w/z)A(w)A(z)$. Since the YBE preserves the partition function at every step, they are equal.

Lecture B2 Exercises. For Problems 1 and 2, use the parametrized version of the six vertex model weights given in lecture.

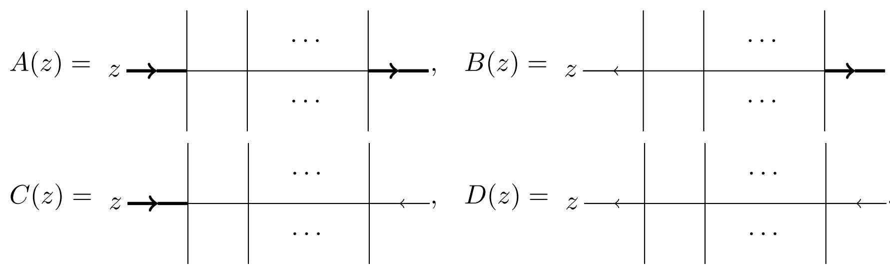
Problem 1: Unitary Relation
Show that the R-matrix satisfies the relation

$$R_{21}(w/z)R_{12}(z/w) = a(z/w)a(w/z)I$$

where I is the identity. Graphically this relation is drawn as



Problem 2: RTT relations
Consider a row of N vertices and row operators



Each of these can be seen as a $2^N \times 2^N$ matrix. Use the Yang-Baxter equation to show that they satisfy the following relations:

$$\begin{aligned}
A(w)A(z) &= A(z)A(w)\\
C(w)C(z) &= C(z)C(w)\\
a(z/w)C(w)A(z) &= c(z/w)C(z)A(w) + b(z/w)A(z)C(w)\\
c(z/w)C(w)D(z) + b(z/w)D(w)C(z) &= a(z/w)C(z)D(w)\\
b(z/w)B(w)C(z) + c(z/w)A(w)D(z) &= b(z/w)C(z)B(w) + c(z/w)A(z)D(w)
\end{aligned}$$

(This is just a subset of the full set of RTT relations.)

Problem 3: Solving the YBE for the six vertex model
Recall the Yang-Baxter equation

$$R_{23}R'_{13}R''_{12} = R''_{12}R'_{13}R_{23}$$

with

$$R = \begin{pmatrix} a & 0 & 0 & 0\\ 0 & b & c & 0\\ 0 & c & b & 0\\ 0 & 0 & 0 & a \end{pmatrix}$$

and similar formulas for R' and R''.

(a) Show that of the 64 equations, due to the ice rule, only 20 are non-trivial.
(b) Show that these equations come in 10 identical pairs by considering what happens after reversing all the arrows.

(c) Note the rotating both diagrams 180 degrees takes the LHS to the RHS. Show that there are 4 rotational symmetric boundary conditions whose equations are automatically satisfied and that the remaining 6 equations come in identical pairs.

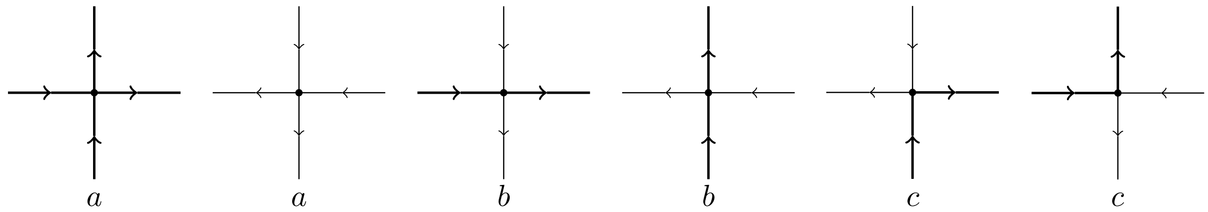
(d) Write out the 3 equations. By attempting to solve for a'', b'', c'' show that a solution exists only if $\Delta = \Delta'$, where
$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$

(e) Show that if $\Delta = \Delta'$, we must have $\Delta = \Delta''$ as well.

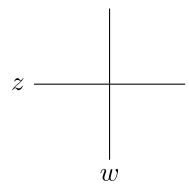
(f) Show that if we choose to parametrize our weight by
$$a(z) = zq - \frac{1}{zq}, \quad b(z) = z - \frac{1}{z}, \quad c(z) = q - \frac{1}{q}$$
and similarly for the primed weights with z', then the double primed weights take the same form with $z'' = z/z'$.

Lecture B3: Domain Wall Boundary Conditions

Last Time:
Recall our 6V model and weights:



We will use the below parametrization

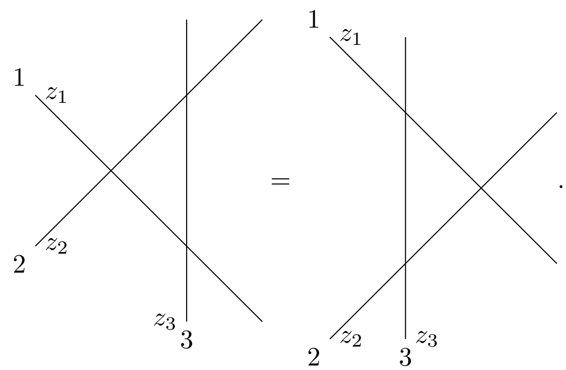


$$\begin{aligned} a(z/w) &= \frac{zq}{w} - \frac{w}{zq} \\ b(z/w) &= \frac{z}{w} - \frac{w}{z} \\ c(z/w) &= q - \frac{1}{q} \end{aligned}$$

with which we have the YBE

$$R_{12}(z_1/z_2)R_{13}(z_1/z_3)R_{23}(z_2/z_3) = R_{23}(z_2/z_3)R_{13}(z_1/z_3)R_{12}(z_1/z_2)$$

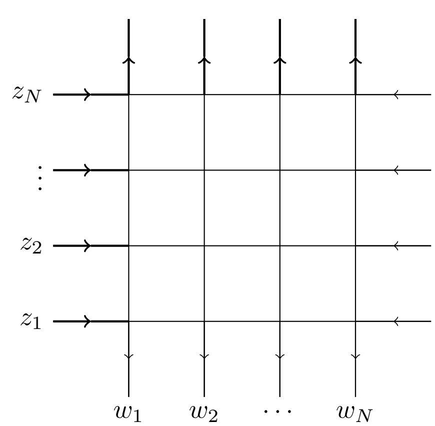
or in graphical form



Domain Wall Boundary Conditions:
Consider an $N \times N$ square with boundary condition given by

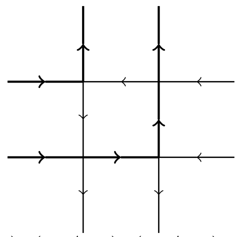
- arrows point inward at the left/right boundaries and outward at the top/bottom boundaries
- or, in other words, paths enter at every row to the right and exit at every column at the top, with all other boundary edges having no path.

We assign a spectral parameter z_i to the i^{th} row and w_i to the i^{th} column as shown below:



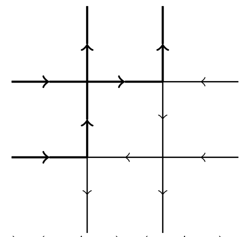
This means that the vertex at row i and column j uses $\frac{z_i}{w_j}$ as the parameter in it weight. Denote the partition function by $Z_N(z_1, \ldots, z_N; w_1, \ldots, w_N)$.

As an example consider the $N = 2$ case. The two possible configurations and their weight are given by



$b(z_1/w_1)c(z_1/w_2)c(z_2/w_1)b(z_2/w_2)$

and



$c(z_1/w_1)a(z_1/w_2)a(z_2/w_1)c(z_2/w_2)$

The partition function is then

$$Z_2(z_2, z_2; w_1, w_2) = b(z_1/w_1)c(z_1/w_2)c(z_2/w_1)b(z_2/w_2) + c(z_1/w_1)a(z_1/w_2)a(z_2/w_1)c(z_2/w_2).$$

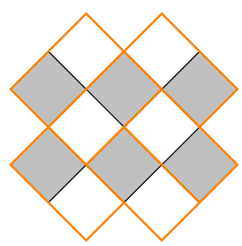
For large N the partition function will be an extremely large, apriori, complicated sum of terms. Amazingly though it has a simple closed formula due to Izergen (1987) expanding on work of Korepin (1982).

Theorem 0.3 (Izergen-Korepin Determinant formula)**.**

$$Z_N(z_1, \ldots, z_N; w_1, \ldots, w_N) = (-1)^{N(N-1)/2} \frac{\prod_{i,j=1}^{N} a(z_j/w_i)b(z_j/w_i)}{\prod_{1 \le i < j \le N} b(z_i/z_j)b(w_i/w_j)} \times \det\left(\frac{c(z_j/w_i)}{a(z_j/w_i)b(z_j/w_i)}\right)_{i,j=1,\ldots,N}.$$

Before we discuss the proof, let us see some connections with combinatorics.

Domino Tilings of the Aztec Diamond:



Above is an example of a domino tiling of the Aztec diamond of rank 2. (Rank indicates how many corners appear at each side of the diamond.)

These tilings were enumerated by Elkies-Kuperberg-Larsen-Propp (1992).

Theorem 0.4. *The number of tilings of the Aztec diamond of rank N is given by*

$$AD_N = 2^{(N+1)N/2}.$$

(Note: they were interested in enumerating alternating sign matrices, which we will discuss next.)

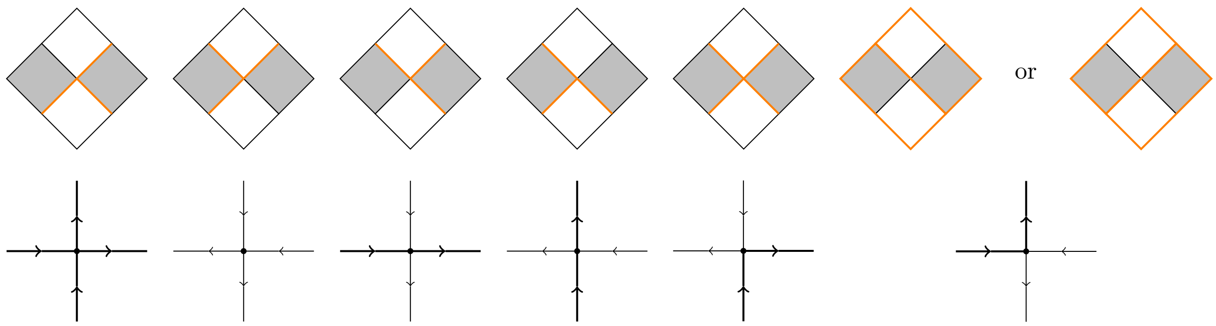
One way to prove this (not the original proof) is by relating the tiling to something we already know how to count, in this case, the domain wall boundary condition six vertex model.

Proposition 0.5.

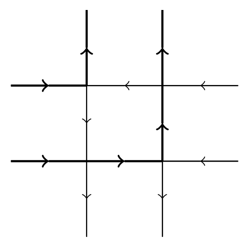
$$AD_N = (\sqrt{2})^N Z_N$$

where Z_N is the partition function of the domain wall boundary condition six vertex model of rank N with weights $a = b = 1$ and $c = \sqrt{2}$. (With these weights, we have $\Delta = 0$. This is known as the "free fermionic point".)

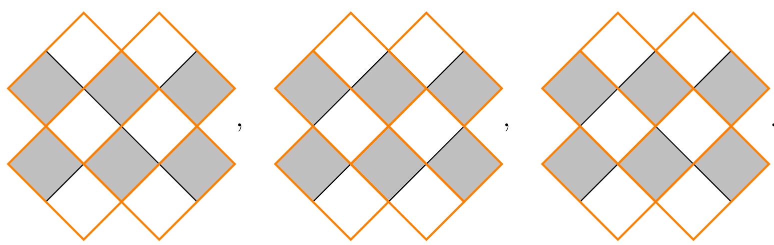
The proof of the proposition follows from a mapping between the vertices of the vertex models and local configurations of the dominos in the tilings. More precisely, the mapping from local domino configurations to vertices is given by



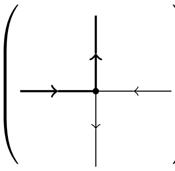
I leave it to the reader to check that this always takes tilings of the AD to domain wall boundary conditions. Note this is a many-to-one mapping in that multiple domino tilings can map to the same six vertex configuration. For example, the tiling of the rank-2 AD above maps to



but so do the tilings you get by rotating pairs of adjacent parallel dominos (in a 2×2 square w/ black square on the left/right)

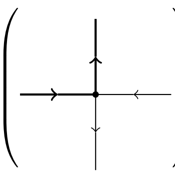
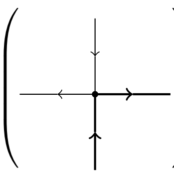


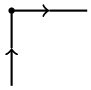
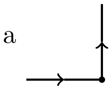
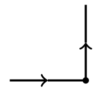
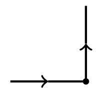
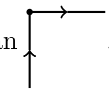
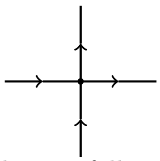
In order to use the six vertex model to count the tilings, we would like to assign weights such that every vertex has weight 1 except

$w\Bigg($  $\Bigg) = 2.$

Unfortunately, this breaks the symmetry in how we have written the weights. (This is not a big issue, and one can consider asymmetric weights, but we won't consider that here.) To overcome this, we will split the weight between the two c-vertices and use the following lemma.

Lemma 0.6. *In any configuration of the DWBC six vertx model or rank N, we have*

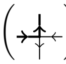
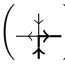
$\#\Bigg($  $\Bigg) = \#\Bigg($  $\Bigg) + N.$

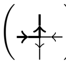
Proof.
- For each a path a  must eventually be followed by a  since the paths must end at the top.
- The first corner of each path is of the form  since path start on the left. Coupled with the above bullet point this means each path has one more  than  .
- Any  removes one of both types of c-vertices, so does not effect the difference.
- The lemma follows since there are N total paths.

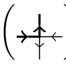
□

Note: while the above is specific to the domain wall boundary conditions, for any choice of fixed boundary conditions there will be a similar relationship between the number of the two types of c-vertices.

To finish the proof of the proposition we have

$(\sqrt{2})^N Z_N = \sum_{\text{configs. } \mathcal{C}} (\sqrt{2})^N c^{\#\left(\right.}$  $^{\left.\right)} c^{\#\left(\right.}$  $^{\left.\right)}$

$= \sum_{\text{configs. } \mathcal{C}} (\sqrt{2})^N c^{2\#\left(\right.}$  $^{\left.\right)} c^{-N}$

$= \sum_{\text{configs. } \mathcal{C}} 2^{\#\left(\right.}$  $^{\left.\right)}$

$= AD_N$

where the last equality follows from our mapping.

Alternating Sign Matrices:
An alterrnating sign matrix of size N is

- an $N \times N$ matrix
- with entries in $\{-1, 0, 1\}$
- such that, if we ignore the zeros, the 1's and -1's alternate along each row and each column
- and, still ignoring zeros, each row and column begins and ends with a 1.

For example, with $N = 3$, we the alternating sign matrices include

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that the first matrix above is a permutation matrix and all other permutation matrices are also alternating sign matrices, but the alternating sign matrices contain more.

Theorem 0.7. *The number of alternating sign matrices of size N is given by*

$$ASM_N = \prod_{i=0}^{N} \frac{(3i+1)!}{(N+i)!}.$$

A bit of history:

- Alternating sign matrices appear when studying a certain method of computing determinants known as Dodgeson condensation. (Due to Charles Dodgeson, aka Lewis Carol, in the 1860s.)
- Mills-Robin-Rumsey conjectured in 1983 that $ASM_N = \prod_{i=0}^{N} \frac{(3i+1)!}{(N+i)!}$.
- Zeilberger (1992-1994) gave a famously long and complicatted proof.
- Kuperberg (1995) gave a much simpler proof using the DWBC six vertex model.

We will see how to relate the alternating sign matrices to six vertex configurations.

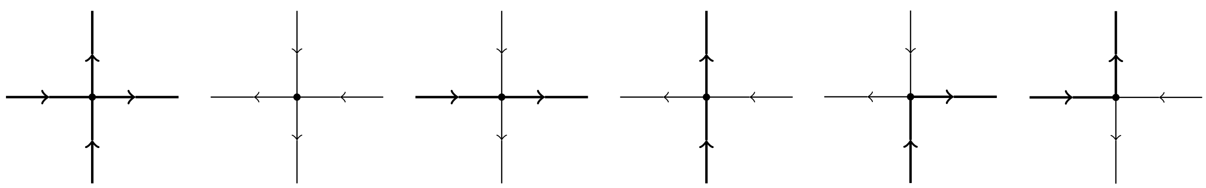
Proposition 0.8. *The number of alternating sign matrices of size N is given by*

$$ASM_N = Z_N$$

where Z_N is the partition function of the DWBC six vertex model or rank N with weights $a = b = c = 1$. (With these weights, we have $\Delta = 1/2$. This is known as the "combinatorial point".)

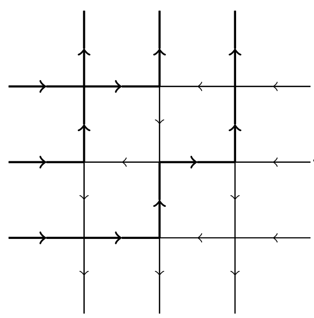
There is a direct bijection between the ASMs and six vertex configurations given by mapping matrix entries to vertices according to the rule:

0 0 0 0 -1 1



For example, we have

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mapsto$$



We need to check that the mapping from the vertex model to the matrix always results in alternating signs. Let us consider a single row since columns are similar. The idea is similar to the argument we used for relating the number of each type of c-vertex.

- Since paths enter on the right the first corner must be of the type corresponding to 1's in the matrix.
- Every corner corresponding to a -1 must be followed by a corner corresponding to a 1 since the paths must end going up.
- For the same reason the last corner will be the kind corresponding to a 1.
- One can check that fully packed a-vertices do not change the above conclusions.

Since we have a bijection between ASMs and DWBC six vertex configurations, we should choose uniform weights for the six vertex model to enumerate ASMs.

Note: One cannot directly apply the Izergen-Korepin determinant formula to enumerate the number of ASMs. When choosing z_i's, w_i's, and q to make $a = b = c$, one will find that when directly plugging these values into the formula, many of the factors in both the numerator and denominator will go to zero. So a slightly technical argument must be done in which the weights start off non-uniform, the determinant is evaluated to be a product, then the limit to uniform weights is taken. This is what Kuperberg does in his proof.

Sketch of the proof of the Izergen-Korepin Determinant formula:

Here will sketch the proof of Izergen-Korepin determinant formula

$$Z_N(z_1,\dots,z_N;w_1,\dots,w_N) = (-1)^{N(N-1)/2}\frac{\prod_{i,j=1}^{N} a(z_j/w_i)b(z_j/w_i)}{\prod_{1\le i<j\le N} b(z_i/z_j)b(w_i/w_j)} \times \det\left(\frac{c(z_j/w_i)}{a(z_j/w_i)b(z_j/w_i)}\right)_{i,j=1,\dots,N}.$$

One can check that when $N=1$ both sides are equal to

$$c(z_1/w_1) = q - \frac{1}{q}.$$

We would like to use induction to extend the equality to general N. Our main lemma used to accomplish this is following recursion relation due to Korepin.

Lemma 0.9.

$$Z_N(z_1,\dots,z_N;w_1=z_1,w_2,\dots,w_N) = c(z_1/w_1)\prod_{i=2}^{N} a(z_1/w_i)a(z_i/w_1) \times Z_{N-1}(z_2,\dots,z_N;w_2,\dots,w_N).$$

Suppose both sides of the determinant formula satisfy the above recursion relation, and that both sides are equal for rank $N-1$. Then the above tells us that, at rank N, they are equal at a single point $w_1 = z_1$ but they could still be different when $w_1 \neq z_1$ (that is, they could differ by terms proportional to $z_1 - w_1$). So we need a little bit more.

Lemma 0.10.

$$Z_N(z_1,\dots,z_N;w_1,\dots,w_N)$$

is symmetric in the $\{z_i\}$ *and in the* $\{w_i\}$.

Lemma 0.10 tells us that in fact both sides of the determinant formula are equal at N different points $w_1 = z_1,\dots,z_N$ since we get that the recursive formula in Lemma 0.9 still holds after permuting the indices in the z_i.

Lemma 0.11.

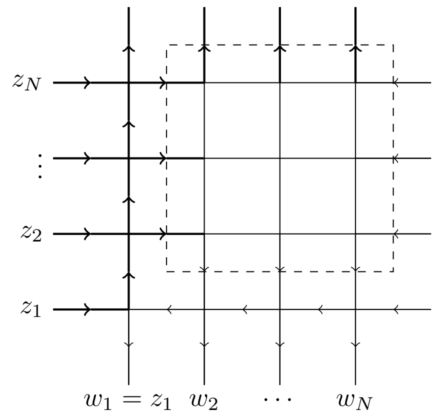
$$w_1^{N-1}Z_N(z_1,\dots,z_N;w_1,\dots,w_N)$$

is a polynomial of degree $N-1$ *in* w_1^2 *(with coefficients that are rational functions in the other parameters).*

Combining this we the previous lemmas, we see that both sides of the determinant formula multiplied by w_1^{N-1} are polynomials of degree $N-1$ in $s = w_1^2$ and agree at the N points $s = z_1^2,\dots,z_N^2$. This uniquely determines the polynomials, so the two sides are equal. (One can see this from Lagrange interpolation, for example.)

So the main theorem is proved if we can show both sides satisfy the above three lemmas. We will give proofs of the lemmas for the partition function and leave the determinant side to the problem set.

Proof of Lemma 0.9. Note that with $w_1 = z_1$, the b-weight for the bottom-left vertex is now $b(1) = 0$. This means the bottom-left vertex must be a c-vertex with the corresponding path turning up. This forces the rest of the configuration in the first row and column as shown below:



The vertices forced by setting $w_1 = z_1$ have weight

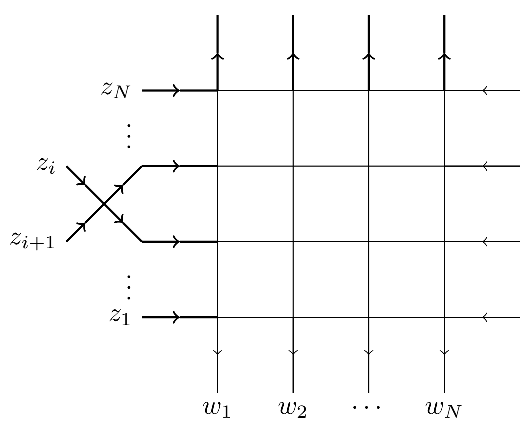
$$c(z_1/w_1)\prod_{i=2}^{N} a(z_1/w_i)a(z_i/w_1)$$

while the rest of the vertices (inside the dashed square) form a domain wall boundary condition six vertex model of size one smaller and contributes

$$Z_{N-1}(z_2, \dots, z_N; w_2, \dots, w_N).$$

□

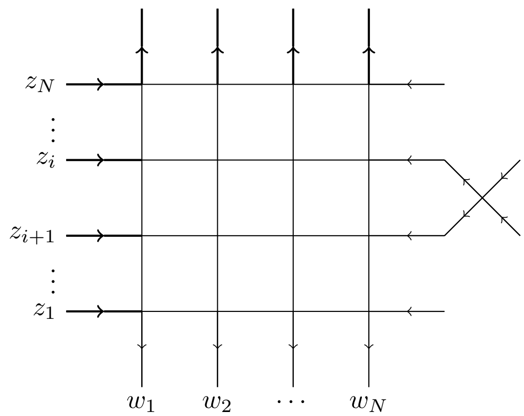
Proof of Lemma 0.10. First, let us show the symmetry in the $\{z_i\}$. The argument is essentially the same as showing the RTT relations in the previous problem set, but we reproduce it here. Suppose we insert a cross vertex connecting rows i and $i+1$ as shown below.



Note that with two paths entering the cross vertex from the left means that two paths are forced to exit on the right. The above has partition function

$$a(z_i/z_{i+1})Z_N(\dots, z_i, z_{i+1}, \dots; w_1, \dots, w_N).$$

By repeatedly using the YBE we can push the cross all the way to the right. Since no paths can exit the cross to the right, we must have no paths entering on the left. The picture is

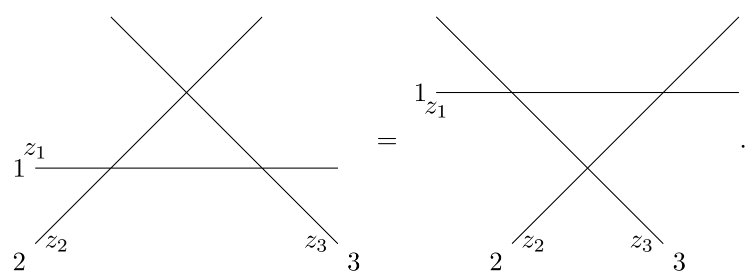


which has partition function

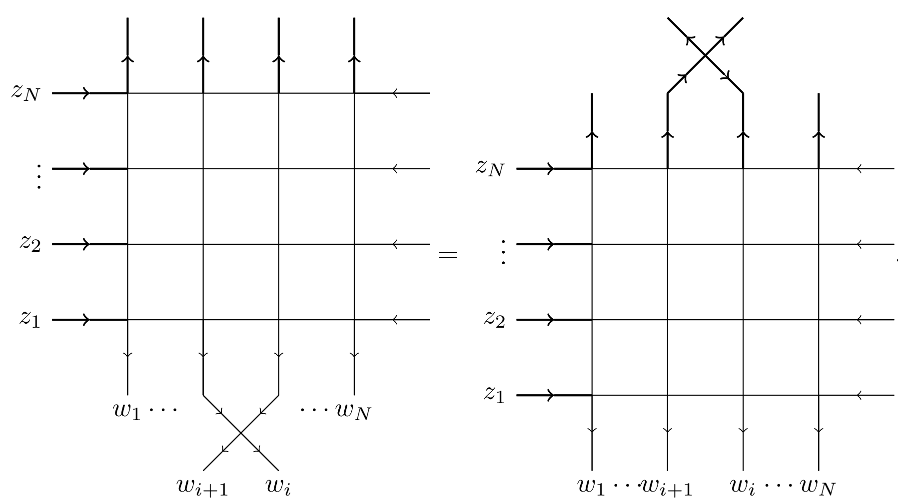
$$a(z_i/z_{i+1})Z_N(\ldots, z_{i+1}, z_i, \ldots; w_1, \ldots, w_N).$$

Since the YBE preserves partition function, these must be equal. Thus we see we can swap z_i and z_{i+1} without changing the partition function.

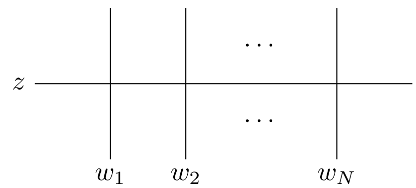
The argument showing the symmetry in the $\{w_i\}$ is similar. We rearrange the strands in the YBE in order to get a column version of the YBE



Then the symmetry follows from the equality



Note, we can generalize the notation from the previous problem set and define a row operator

$C(z; w_1, \dots, w_N) =$ 

that now includes column parameters. Then the symmetry in the $\{z_i\}$ follows from the RTT relation saying that the C operators commute. $\square$

Proof of Lemma 0.11. Note that w_1 only appears in the first column and that in the first column there will be exactly one c-vertex. This means there are $N-1$ a- and b-vertices whose weight is either

$$\frac{z_i q}{w_1} - \frac{w_1}{z_i q} \quad \text{or} \quad \frac{z_i}{w_1} - \frac{w_1}{z_i}$$

and this exhaust the appearances of w_1 in the partition function. When we multiply the partition function by w_1^{N-1} and distribute a w_i to each of the to a- and b-vertex weights they become

$$z_i q - \frac{w_1^2}{z_i q} \quad \text{or} \quad z_i - \frac{w_1^2}{z_i}.$$

We see that everything is now a polynomial in $s = w_1^2$ and that the highest degree term in s has degree $N-1$, as desired. $\square$

Lecture B3 Exercises.

Problem 1: Uniform Weights and ASMs
In lecture we showed that the number of alternating size matrices of size N is equal to the number six vertex configurations with domain wall boundary conditions of rank N and weights $a = b = c = 1$. We would like to relate that to our parametrization of the weights

$$a(z/w) = \frac{zq}{w} - \frac{w}{zq}, \quad b(z/w) = \frac{z}{w} - \frac{w}{z}, \quad c(z/w) = q - \frac{1}{q}.$$

(a) Find values of q, z, w such that $a = b = c \neq 0$.
(b) With this choice of parametrization, find β such that

$$Z_N(z, \ldots, z; w, \ldots, w) = \beta \times \# \text{ of ASMs of size } N.$$

Problem 2: Strict Gelfand-Tsetlin patterns
A strict Gelfand-Tsetlin pattern is a triangular array of non-negative integers

$$\begin{array}{ccccccc} x_{11} & & x_{12} & & \cdots & & x_{1n} \\ & x_{21} & & \cdots & & x_{2,n-1} & \\ & & \ddots & \cdots & ⋰ & & \\ & & & x_{n1} & & & \end{array}$$

such that the entries are strictly decreasing along rows, weakly decreasing down NW-SE diagonals, and weakly increasing down NE-SW diagonals.

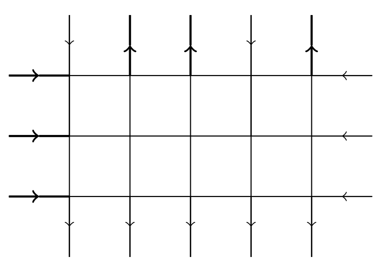
Show that strict Gelfand-Tsetlin patterns with top row given by

$$\lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_N \geq 0)$$

are in bijection with six vertex model configurations on an N row by $\lambda_1 + 1$ column grid such that

- a path enters at each row on the left,
- for each i a path exits at column λ_i on top (indexing from left to right starting at 0),
- and all other boundary edges are empty.

As an example of the boundary conditions, if $\lambda = (4, 2, 1)$ we have



Note that domain wall boundary conditions correspond to the case

$$\lambda = (N - 1, N - 2, \ldots, 0).$$

Problem 3: Izergen-Korepin determinant
Recall the Izergen-Korepin determinant formula from lecture

$$Z_N(z_1, \ldots, z_N; w_1, \ldots, w_N) =$$
$$(-1)^{N(N-1)/2} \frac{\prod_{i,j=1}^{N} a(z_j/w_i) b(z_j/w_i)}{\prod_{1 \leq i < j \leq N} b(z_i/z_j) b(w_i/w_j)} \det \left(\frac{c(z_j/w_i)}{a(z_j/w_i) b(z_j/w_i)} \right)_{i,j=1,2,\ldots,N}.$$

Show that the RHS:

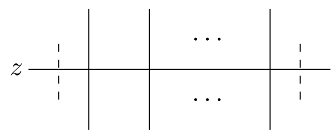
(a) is symmetric in the $\{z_i\}$ and in the $\{w_i\}$.

(b) satisfies

$$Z_N(z_1,\ldots,z_N;w_1=z_1,w_2,\ldots,w_N)=$$
$$c(z_1/w_1)\left(\prod_{i=2}^{N}a(z_1/w_i)a(z_i/w_1)\right)Z_{N-1}(z_2,\ldots,z_N;w_2,\ldots,w_N).$$

Problem 4: Algebraic Bethe Ansatz

Consider a row of N vertices on a cylinder and let $\tau(z)=A(z)+D(z)$ be the corresponding row operator (using the notation from the previous problem set). Graphically, we have

$\tau(z) =$ 

where the edges crossing the dashed lines are identified. In this exercise, we want to find the eigenvectors and eigenvalues of this $2^N\times 2^N$ matrix.

(a) Let $\vec{v}_0=e_-\otimes\ldots\otimes e_-$. Show that it is the an eigenvector of $\tau(z)$ with eigenvalue $\Lambda_0=b(z)^N+a(z)^N$. (Hint: first show it is an eigenvector of both $A(z)$ and $D(z)$.)

Now let $\vec{v}_1=C(w)\vec{v}_0$. We will show that for appropriate choices of w, $\vec{v}_1$ is also an eigenvector.

(b) Using the RTT relations from the previous problem set, show that

$$\begin{aligned}A(z)\vec{v}_1 &= \left(\frac{a(z/w)}{b(z/w)}C(w)A(z)-\frac{c(z/w)}{b(z/w)}C(z)A(w)\right)\vec{v}_0\\ D(z)\vec{v}_1 &= \left(\frac{a(w/z)}{b(w/z)}C(w)D(z)-\frac{c(w/z)}{b(w/z)}C(z)D(w)\right)\vec{v}_0\end{aligned}.$$

(c) Show that if w satisfies

$$\left(\frac{b(w)}{a(w)}\right)^N=-\frac{b(z/w)}{c(z/w)}\frac{c(w/z)}{b(w/z)}$$

then $\vec{v}_1$ is an eigenvector of $\tau(z)$ with eigenvalue

$$\Lambda_1(z;w)=b(z)^N\frac{a(z/w)}{b(z/w)}+a(z)^N\frac{a(w/z)}{b(w/z)}.$$

In general, $\vec{v}_n=C(w_n)\ldots C(w_2)C(w_1)\vec{v}_0$ is an eigenvector of $\tau(z)$ if the parameters $w_1,\ldots,w_n$ satisfy the n coupled equations

$$\left(\frac{b(w_k)}{a(w_k)}\right)^N=-\frac{b(z/w_k)}{c(z/w_k)}\frac{c(w_k/z)}{b(w_k/z)}\prod_{\substack{\ell=1\\ \ell\neq k}}^{n}\frac{a(w_\ell/w_k)}{b(w_\ell/w_k)}\frac{b(w_k/w_\ell)}{a(w_k/w_\ell)}$$

for each $k=1,2,\ldots,n$. These are known as the *Bethe equations.* The corresponding eigenvalue is

$$\Lambda_n(z;w_1,\ldots,w_n)=b(z)^N\prod_{k=1}^{n}\frac{a(z/w_k)}{b(z/w_k)}+a(z)^N\prod_{k=1}^{n}\frac{a(w_k/z)}{b(w_k/z)}.$$

(You do not need to show this, but feel free to try if you're feeling ambitious. In the general case, it is helpful to remember that the C operators commute.)

(d) Recall that in our construction here each of the row operators has N vertices. What can you say about how the vector $\vec{v}_N$ relates to DWBCs?

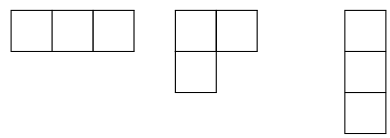
Lecture B5: Five Vertex Models and Schur Polynomials

Schur Polynomials:
Let n be a positive integer. An *integer partition* of n (often just called a partition of n) is a way to write n as a sum of non-negative integers. For example, one partition of $n = 3$ is $3 = 2 + 1$. We do not distinguish between different rearrangements of the summands, that is, we treat $2 + 1$ and $1 + 2$ as the same partition. If λ is a partition of n, we usually write it as a tuple of the summands put in decreasing order:

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$$

with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ and $\lambda_1 + \ldots + \lambda_k = n$.

We will often depict partitions pictorially in the form of *Young diagrams.* These are collections of boxes such that the number of boxes in the ith row is equal to the ith part of the partition. It is easiest to see in an example. For the partitions of 3, the Young diagrams are



$\lambda = (3) \qquad \lambda = (2,1) \quad \lambda = (1,1,1)$

Note that we are drawing the largest part as the top row. This is the *English convention.* The *French convention* has the largest part on bottom and the smallest on top.

The *Schur polynomials* $s_\lambda(x_1, x_2, \ldots, x_k)$ are a family of symmetric polynomials in the variables $x_1, \ldots, x_k$ indexed by integer partitions. One way to define them is through the bialternant formula:

$$s_\lambda(x_1, \ldots, x_k) = \frac{1}{\prod_{1 \leq i < j \leq k}(x_i - x_j)} \det \left(x_j^{\lambda_i + k - i} \right)_{i,j=1,\ldots,k}.$$

An alternate definition is through semi-standard Young tableaux. A *semi-standard Young tableaux* of shape λ is a filling of the cells of the Young diagram of λ by entries in $\{1, 2, \ldots, n\}$ such that:

- The entries are weakly increasing across the rows.
- The entries are strictly increasing down the columns.

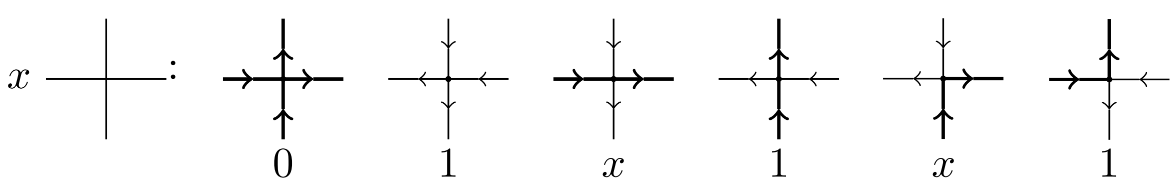
Let $SSYT(\lambda)$ be the set of SSYT of shape λ. The Schur polynomial can be written

$$s_\lambda(x_1, \ldots, x_k) = \sum_{T \in SSYT(\lambda)} x^T$$

where $x^T = x_1^{\#\text{ of 1's in } T} \ldots x_k^{\#\text{ of } k\text{'s in } T}$.

A First Five-Vertex Model:

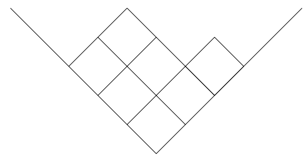
We would like to give a vertex model whose partition function give the Schur polynomials. First, we need our vertices and their weights:



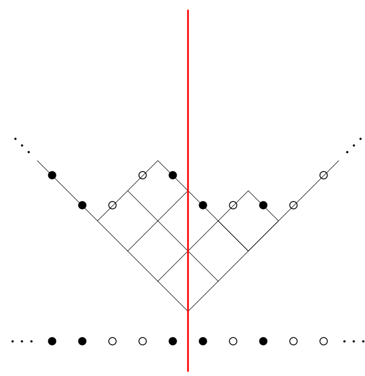
Note that there are now only five allowed vertices (the fully packed vertex has its weight set to zero), so we call this a five-vertex model. In matrix form we have

$$L(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & x & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Next, we want a domain and boundary conditions. To relate the vertex model to the Schur polynomials, it will be useful for us to have another perspective through which we can view partitions. Consider the Young diagram of a partition λ drawn in *Russian convention* (French convention rotated 45° counter-clockwise). For example, if $\lambda = (3, 2, 2)$ we draw



The boundary of the partition then becomes an path with steps of the form $(1, 1)$ and $(1, -1)$. Assign to each $(1, -1)$ step a particle (or filled dot) and to each $(1, 1)$ step a hole (or unfilled dot). Project this sequence of particles and holes to the horizontal axis, and extend it infinitely to the left with particles and infinitely to the right with holes. This sequence of particles and holes is known as the *Maya diagram* of the partition. In our example, we have

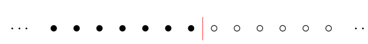


Note there is a unique point where the number of particles to the right of that point is equal to the number of holes to the left. We call this *center* of the Maya diagram and draw it in red in the above. We will think of the center being at 0 and the particles as being on the half-integers. In the example above the particle position, starting with the right-most, are given by

$$(2.5,\ 0.5,\ -0.5,\ -3.5,\ -4.5,\ \ldots)$$

In fact, if $\lambda = (\lambda_1, \lambda_2, \ldots)$, including infinitely many zero parts, then the particles are at positions $\lambda_i - i + \frac{1}{2}$ for $i = 1, 2, \ldots$.

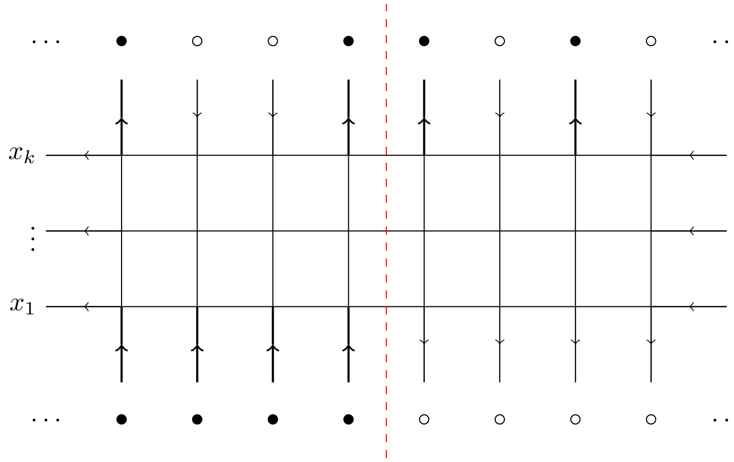
The Maya diagram of the empty partition $\emptyset$ will appear frequently. For this partition the particles are fully-packed to the left of center and the holes to the right:



While we consider Maya diagrams as bi-infinite sequences of particles and holes, notice that for any given partition λ there is only a finite interval in which anything "interesting" happens. If λ_1 is the length of the first row of λ and λ'_1 the length of the first column, then the right-most particle is at $\lambda_1 - \frac{1}{2}$ and the left-most hole is at $-\lambda'_1 + \frac{1}{2}$.

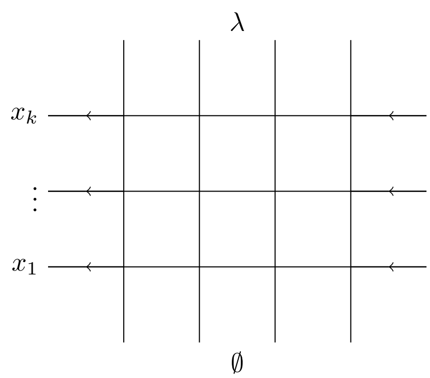
Note that to recover the partition from the Maya diagram one does the following: count how many spaces the i^{th} particle from the right would have to jump to get from its position in the empty partition to its current position. This is equal to the i^{th} part of the partition.

We will use the Maya diagrams to assign the boundary conditions to our domains. As an example, for $\lambda = (3, 2, 2)$, our domain would look like



- The vertex model is made of bi-infinite rows.
- The top boundary is determined by the Maya diagram for λ, that is, a particle in the Maya diagram corresponds to a boundary path.
- The bottom boundary is determined by the Maya diagram for the empty partition. In particular, the paths are fully packed to the left.
- We mark the center in red so that we know how to align the Maya diagrams on the rows of the vertex model.
- No paths enter or exit on the sides. Note this means that sufficiently far to the right all vertices are forced to be empty, while sufficiently far to the left all vertices are forced to contain vertical paths. These vertices have weight 1 and do not effect the weight of a configuration.
- Paths have freedom only in columns corresponding to the "interesting" part of the Maya diagrams. Sometimes we will restrict ourselves only to this finite domain.

In general, we will draw pictures of the form



where labeling a boundary by λ means that positions of paths at that boundary are determined by the Maya diagram for λ.

Let $Z_\lambda(x_1, \ldots, x_k)$ be the partition function.

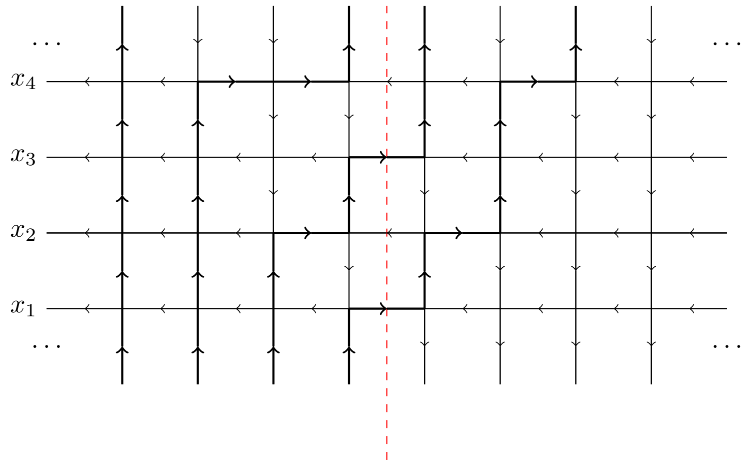
Proposition 0.12.

$$Z_\lambda(x_1, \ldots, x_k) = s_\lambda(x_1, \ldots, x_k)$$

One way to prove this is to give a weight-preserving bijection between vertex configurations and SSYT. Let's see this in an example. Consider the SSYT of shape $\lambda = (3, 2, 2)$ given by

1	2	4
2	3	
4	4	

This maps to the vertex configuration

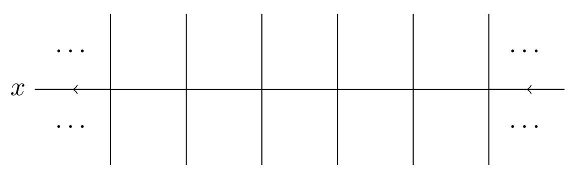


.

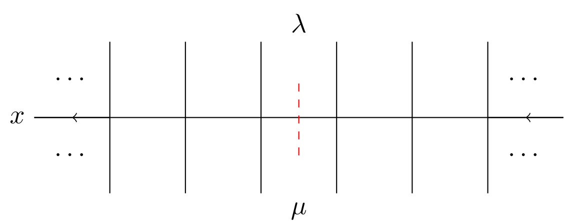
- The i^{th} row of the tableaux determines the i^{th} path from the right.
- The entries in the tableaux determine the row of the vertex model in which a path exits a vertex to the right.
- Empty rows of the tableaux correspond to paths going only vertical.
- Note that fact the the paths only go up and right enforces the weakly increasing condition for the rows of the tableaux, while the fact that the fully packed vertex has weight zero enforces the strictly increasing condition along columns of the tableaux.

For example, the top row of the tableaux with entries 1, 2, 4 corresponds to the rightmost path which goes right in rows 1, 2, and 4 of the vertex model. The weight of this vertex model configuration is $x_1 x_2^2 x_3 x_4^3$, exactly in agreement with what appears in the tableaux formulation of the Schur polynomial.

Let us take a closer look at a single row of this vertex model.

$D(x) =$ 

When we specify the positions of the paths entering and exiting the row we have

$e_\lambda^T D(x) e_\mu =$ 

where e_λ is the basis vector that corresponds to the Maya diagram for λ.

The weights of the 5-vertex model result in constraints on which choices of entering and exiting paths have none zero weight, which we will now describe. We say that two partitions μ and λ *interlace* if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots$$

and write $\mu \preceq \lambda$. Equivalently, $\mu \preceq \lambda$ if and only if one can get the Young diagram of λ from the Young diagram of μ by adding at most one cell to each column. We say that the two partitions differ by a *horizontal strip*.

Lemma 0.13. *There is a vertex configuration with nonzero weight for $e_\lambda^T D(x) e_\mu$ iff $\mu \preceq \lambda$. Moreover, the configuration is unique and has weight $x^{|\lambda/\mu|}$ where $|\lambda/\mu|$ is the number of cells in the Young diagram for λ minus the number of cells in the Young diagram for μ.*

The proof of this lemma is left to the exercises. Note that in terms of the row operator, the Schur polynomial can be written

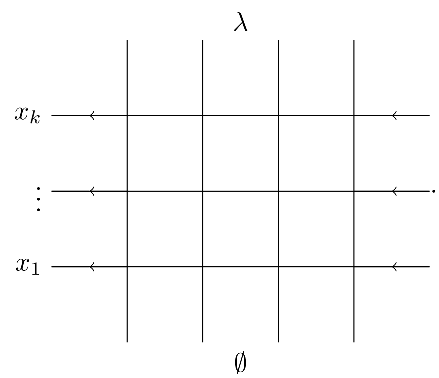
$$s_\lambda(x_1, \ldots, x_k) = e_\lambda^T D(x_k) \cdots D(x_2) D(x_1) e_\emptyset.$$

Using the above lemma we can prove the following branching rule.

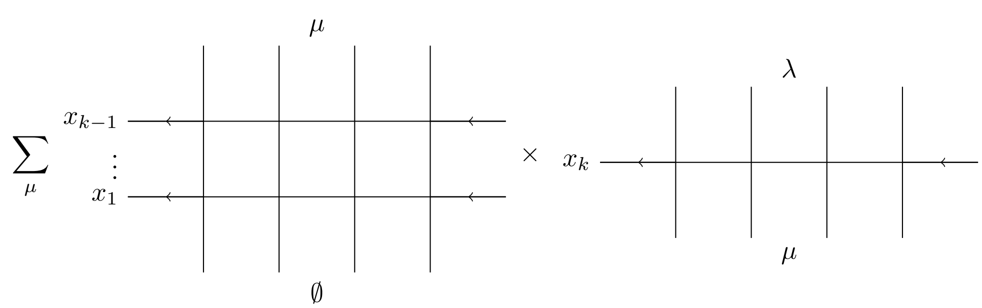
Proposition 0.14 (Branching Rule)**.**

$$s_\lambda(x_1, \ldots, x_k) = \sum_{\mu \preceq \lambda} s_\mu(x_1, \ldots, x_{k-1}) x_k^{|\lambda/\mu|}.$$

Proof. We'll give a pictorial proof. The Schur polynomial $s_\lambda(x_1, \ldots, x_k)$ is given by the partition function



We can split off the top row and consider where the paths entered the top row. Call the partition corresponding to those positions μ. We have



By the previous lemma, the only μ that contribute non-zero terms to the sum are those such that $\mu \preceq \lambda$. For all other μ the weight of the top row is zero. Translating back to Schur polynomials gives the propostition. ☐

We can iterate the branching rule. Set $\lambda^{(k)} = \lambda$ and $\lambda^{(0)} = \emptyset$. Repeatedly replying the branching rule, we have

$$\begin{aligned}
s_\lambda(x_1, \ldots, x_k) &= \sum_{\lambda^{(k-1)} \preceq \lambda^{(k)} = \lambda} s_{\lambda^{(k-1)}}(x_1, \ldots, x_{k-1}) x_k^{|\lambda^{(k)}/\lambda^{(k-1)}|} \\
&= \sum_{\lambda^{(k-2)} \preceq \lambda^{(k-1)} \preceq \lambda^{(k)} = \lambda} s_{\lambda^{(k-2)}}(x_1, \ldots, x_{k-2}) x_{k-1}^{|\lambda^{(k-1)}/\lambda^{(k-2)}|} x_k^{|\lambda^{(k)}/\lambda^{(k-1)}|} \\
&\cdots \\
&= \sum_{\emptyset = \lambda^{(0)} \preceq \cdots \preceq \lambda^{(k-2)} \preceq \lambda^{(k-1)} \preceq \lambda^{(k)} = \lambda} x_1^{|\lambda^{(1)}/\lambda^{(0)}|} \cdots x_{k-1}^{|\lambda^{(k-1)}/\lambda^{(k-2)}|} x_k^{|\lambda^{(k)}/\lambda^{(k-1)}|}.
\end{aligned}$$

In the vertex model, $\lambda^{(i)}$ corresponds to the position at which the path exit row i.

From this we see that there is a bijection between SSYT and sequences of interlacing partitions. (**Note:** This is an example of a Schur process.) Rather than having to translate the SSYT to a vertex model configuration then read off the sequence of partitions by look at the location at which the paths in the vertex model exit each row, one would like to directly see the sequence from the tableaux. We can see how this is done in an example. Consider the SSYT from before:

1	2	4
2	3	
4	4	

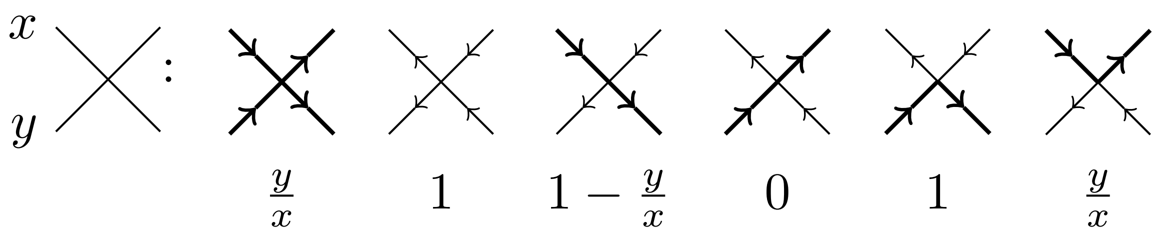
If we look at only the cells which contain values ≤ 1 we see a smaller partition $\lambda^{(1)} = (1)$. Next, we look at all the cells which contain values ≤ 2 and we get $\lambda^{(2)} = (2,1)$. Repeating this gives the sequence

$$\lambda^{(0)} = \emptyset,\ \lambda^{(1)} = (1),\ \lambda^{(2)} = (2,1),\ \lambda^{(3)} = (2,2),\ \lambda^{(4)} = \lambda = (3,2,2).$$

One can check that the inequalities defining the SSYT are equivalent to the interlacing conditions.

Integrability:
So far we have only used the bijection between the vertex model configurations and SSYT. We would look to utilize the tools unique to vertex models; in particular, we want to make use of Yang-Baxter integrability.

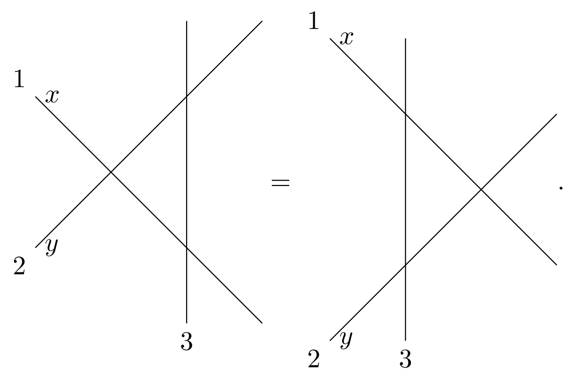
First, we need to define the cross weights for the YBE. They take the form



which we encode in the matrix

$$R(y/x) = \begin{pmatrix} \frac{y}{x} & 0 & 0 & 0 \\ 0 & 1-\frac{y}{x} & 1 & 0 \\ 0 & \frac{y}{x} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The YBE then takes the form



or in matrix form

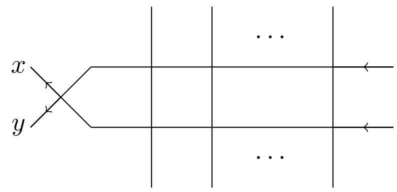
$$L_{23}(y)L_{13}(x)R_{12}(y/x) = R_{12}(y/x)L_{13}(x)L_{23}(y).$$

Note that, unlike in the six vertex model the cross vertex R has a different parametrization than the vertex weights L.

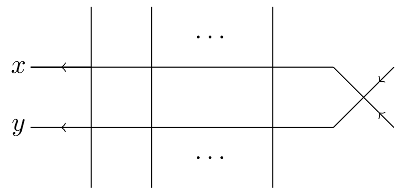
Lemma 0.15.

$$D(y)D(x) = D(x)D(y)$$

Proof. This follows the usual argument. We add an extra "cross" vertex to the left of the row:



which has partition function in $D(y)D(x)$. Now use YBE equation to push the cross from the LHS to the RHS and get



which gives $D(x)D(y)$. $\square$

By writing the Schur polynomial as

$$s_\lambda(x_1, \dots, x_k) = e_\lambda^T D(x_k) \cdots D(x_2) D(x_1) e_\emptyset$$

the previous lemma shows us that we can permute the order of the x_i in the D operators. The symmetry follows from the integrability of the vertex model!

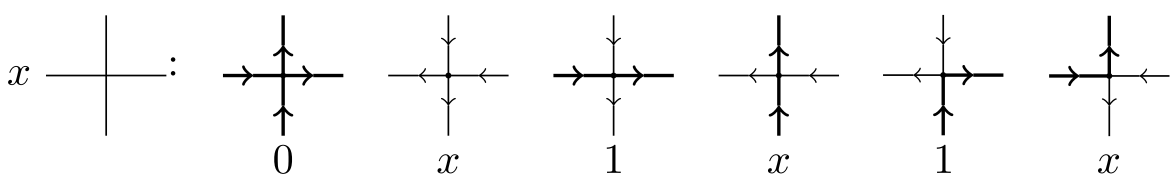
The Cauchy Identity:
The goal for the remainder of this lecture is to prove the Cauchy identity.

Proposition 0.16.

$$\sum_\lambda s_\lambda(x_1, \dots, x_k) s_\lambda(y_1, \dots, y_k) = \prod_{i,j=1}^{k} \frac{1}{1 - x_i y_j}$$

where this should be seen as an equality of power series.

We would like to prove this using vertex model techniques. To do so we will need a second five-vertex model.



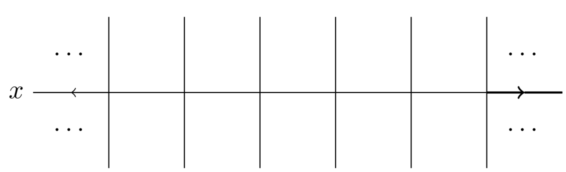
which in matrix form is

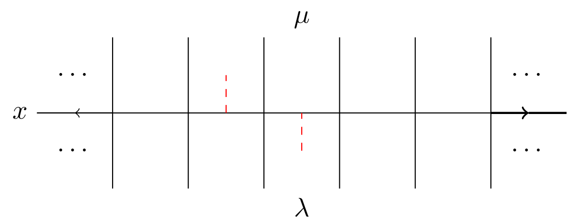
$$L'(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & x & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}.$$

Note that

$$L'(x) = xL(1/x)$$

so this is really only a change of coordinates from our first 5V model. Again let us consider a single row:

$B'(x) =$ 

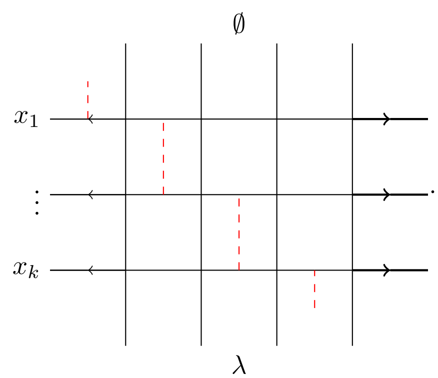
$e_\mu^T B'(x) e_\lambda =$ 

Lemma 0.17. *There is a vertex configuration with nonzero weight for $e_\mu^T B'(x)e_\lambda$ iff $\mu \preceq \lambda$. Moreover, the configuration is unique and has weight $x^\ell x^{|\lambda/\mu|}$ where ℓ is number of paths exiting the row to the top.*

As with the analogous lemma for our original 5V model, we leave the proof to the exercises.

- Note for these weights the larger partition is the bottom boundary and the smaller partition is the top boundary.
- Note that we need to shift where we place the center of the Maya diagram one place to the left on the top compared to the bottom.
- For this type of row, if we allow it to extend infinitely to the left there will be an infinite number of paths exiting att the top, resulting in a factor in the weight x^ℓ with $\ell = \infty$. This can be dealt with by only considering ratios of weights (i.e. always dividing by $e_\emptyset^T B'(x)e_\emptyset$) to cancel the infinite powers. Or by always having the row extend a finite distance to the left. We'll take the latter approach here. We still allow the row to extend infinitely to the right.

Now consider a domain of the form



Call the partition function $Z'_\lambda(x_1, \dots, x_k)$.

- Note the shifting center. We can always assume λ has k parts by adding or removing zero parts. If we restrict how far the domain extends to the left to only the interesting part, then the top boundary will have no paths.

Proposition 0.18.

$$Z'_\lambda(x_1, \dots, x_k) = e_\emptyset^T B'(x_1) \cdots B'(x_k) e_\lambda = x_k^{k-1} \cdots x_2^1 x_1^0 \, s_\lambda(x_1, \dots, x_k).$$

The prefactor $x_k^{k-1} \cdots x_2^1 x_1^0$ comes from the paths exiting the rows towards the top. It is completely determined by the number of paths and does not depend on the configuration. The Schur polynomial comes from the interlacing sequence of partitions, this time going from largest to smallest.

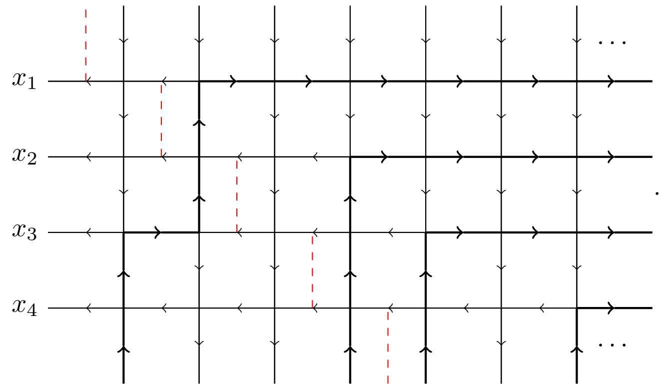
As an example, again consider the SSYT from before:

1	2	4
2	3	
4	4	

corresponding to the sequence

$$\lambda^{(0)} = \emptyset, \ \lambda^{(1)} = (1), \ \lambda^{(2)} = (2,1), \ \lambda^{(3)} = (2,2), \ \lambda^{(4)} = \lambda = (3,2,2).$$

The resulting vertex model configuration is given by



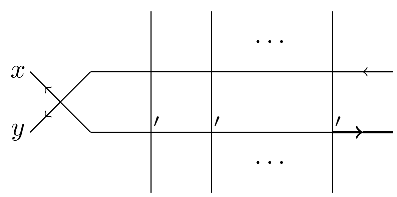
Note that since $L'(x) = xL(1/x)$, the YBE still holds. It now takes the form

$$L_{23}(y)L'_{13}(x)R_{12}(xy) = R_{12}(xy)L'_{13}(x)L_{23}(y).$$

Lemma 0.19. *Suppose* $|x|, |y| < 1$ *then*

$$B'(x)D(y) = \frac{1}{1-xy}D(y)B'(x).$$

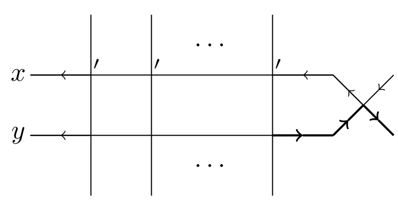
Proof. We start with two rows



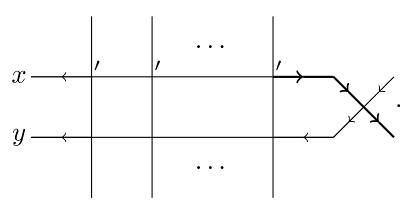
where the primes indicate that those vertices are using the L' weights. These have weight

$$D(y)B'(x).$$

Pushing the cross all the way to the right, we end up with two possibilities



or



Suppose the rows extend far, but not infinitely, to the right. In the top case, the bottom row will have a long sequence of vertices all of which have a path exiting right, while the top row will have a long sequence of empty vertices. These vertices give a weight of $(xy)^M$ for some large M. Letting the rows become infinitely long from the right, we see that $(xy)^M \to 0$ since $|x|, |y| < 1$. Thus the top case has weight zero.

The bottom case contributes and gives

$$(1-xy)B'(x)D(y).$$

Overall we have

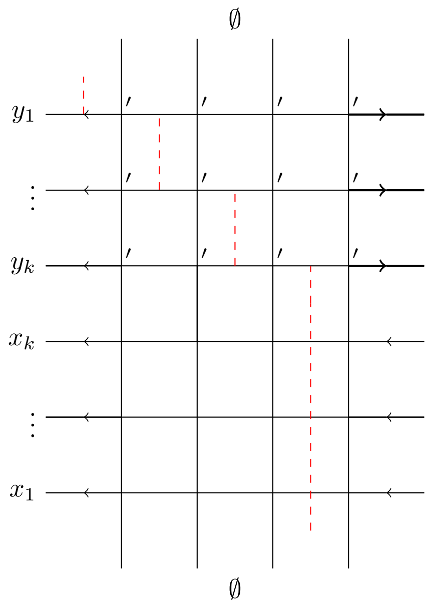
$$D(y)B'(x) = (1-xy)B'(x)D(y)$$

which rearranging finishes the proof. □

We are now ready to prove the Cauchy identity. First, consider the partition function

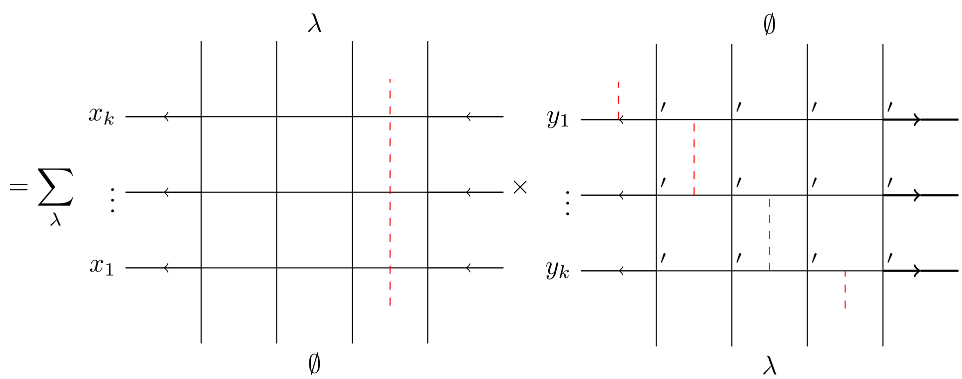
$$e_\emptyset^T B'(y_1)\cdots B'(y_k)D(x_k)\cdots D(x_1)e_\emptyset.$$

Diagrammatically, this looks like



Splitting the top and bottom in the middle, and letting λ indicate where the path exit the k^{th} row, we have

$$e_\emptyset^T B'(y_1)\cdots B'(y_k)D(x_k)\cdots D(x_1)e_\emptyset$$



$$= (y_k^{k-1}\cdots y_2^1 y_1^0)\sum_\lambda s_\lambda(x_1,\dots,x_k)s_\lambda(y_1,\dots,y_k).$$

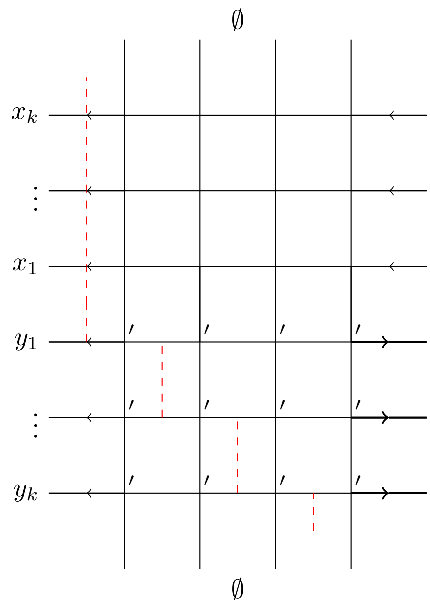
Swapping all the B' operators to the right and D operators to the left using the YBE, we also have

$$e_\emptyset^T B'(y_1)\cdots B'(y_k)D(x_k)\cdots D(x_1)e_\emptyset = \left(\prod_{i,j=1}^k \frac{1}{1-x_iy_j}\right) e_\emptyset^T D(x_k)\cdots D(x_1)B'(y_1)\cdots B'(y_k)e_\emptyset.$$

Viewing the RHS diagrammatically, we have

$$e_\emptyset^T D(x_k)\cdots D(x_1)B'(y_1)\cdots B'(y_k)e_\emptyset$$

$=$



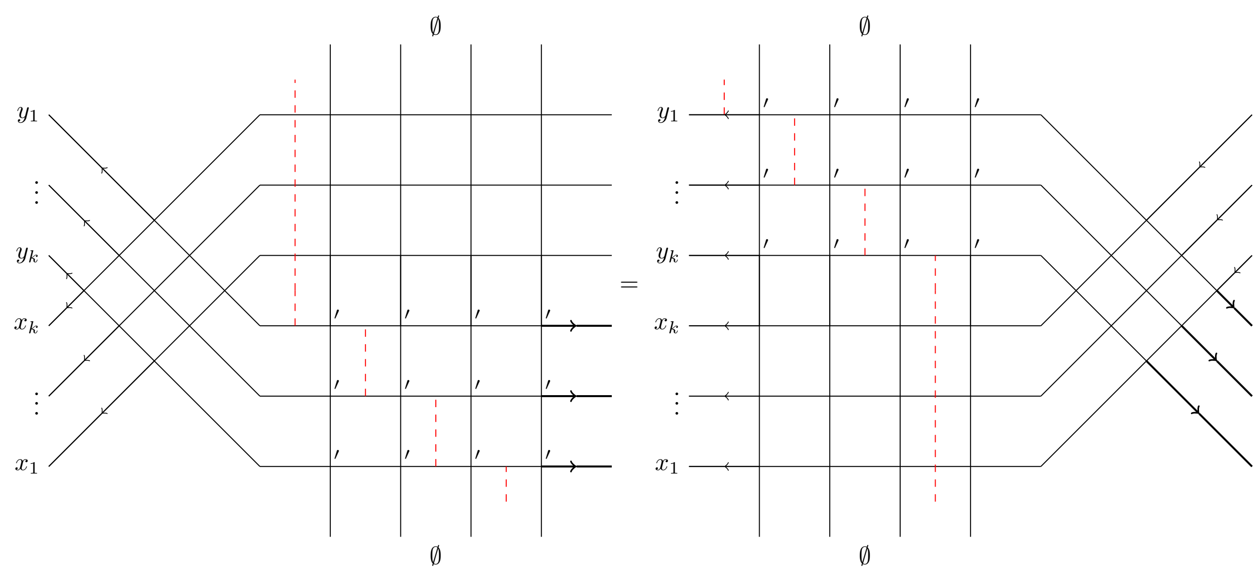
$= (y_k^{k-1}\cdots y_2^1 y_1^0)$

where the last equality follows as it is the weight of the unique configuration with the given boundary conditions. Altogether, we've shown

$$(y_k^{k-1}\cdots y_2^1 y_1^0)\sum_\lambda s_\lambda(x_1,\ldots,x_k)s_\lambda(y_1,\ldots,y_k) = (y_k^{k-1}\cdots y_2^1 y_1^0)\left(\prod_{i,j=1}^{k}\frac{1}{1-x_iy_j}\right)$$

from which the Cauchy identity follows.

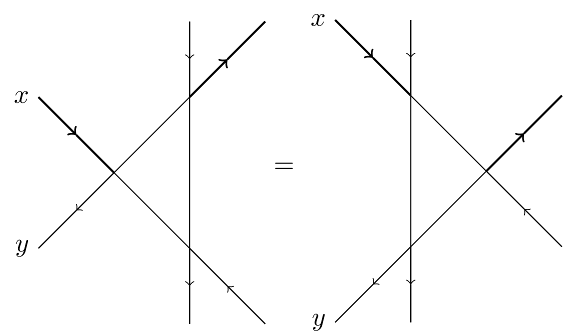
Often the Cauchy identity will be displayed completely diagrammatically as



where one gets from the LHS to the RHS by moving crosses across the domain one at a time, each time swapping a pair of row.

Lecture B5 Exercises.

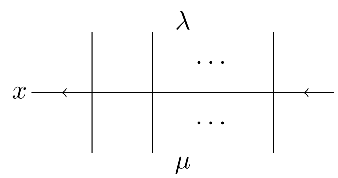
Problem 1: YBE
Check the five vertex Yang-Baxter equation for one choice of boundary conditions:



where we use the L weights (with appropriate parameter) for the columns and $R(y/x)$ for the cross.

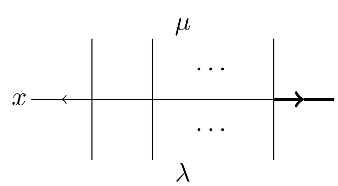
Problem 2: Interlacing Conditions

(a) For a single row using the weights $L(x)$



show that there is only a valid configuration if $\lambda \succeq \mu$ and that the weight is $x^{|\lambda/\mu|}$.

(b) **Optional:** For a single row using the weights $L'(x)$



show that there is only a valid configuration if $\lambda \succeq \mu$ and that the weight is $x^n x^{|\lambda/\mu|}$ where n is the number of paths exiting at the top. (Recall that where you center the Maya diagram for the top boundary is shifted one to the left compared to the bottom.)

Problem 3: Jacobi-Trudi Identity
Consider the five vertex model with weights $L(x)$ from lecture.

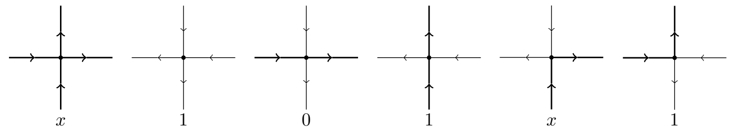
(a) Show that a domain with n rows, with the row i having parameter x_i, that has a single path entering at the bottom in column 0 and exiting at the top in column k has partition function equal to the k^{th} complete homogeneous symmetric polynomial

$$h_k(x_1, \ldots, x_n) = \sum_{1 \le i_1 \le \ldots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

(b) Use part (a) along with our vertex model formulations for the Schur polynomials and the Lindström–Gessel–Viennot lemma to prove the Jacobi-Trudi identity

$$s_\lambda(x_1, \ldots, x_n) = \det\left(h_{\lambda_i+j-i}(x_1, \ldots, x_n)\right)_{i,j=1,\ldots,n}.$$

Problem 4: Another five vertex model
Consider a five vertex model with vertices



Let $\tilde{Z}_\lambda(x_1, x_2, \ldots, x_n)$ be the partition function for for the model with n rows, with parameter x_i in row i, and top boundary determined by λ.

(a) Show that
$$\tilde{Z}_\lambda(x_1, \ldots, x_n) = s_{\lambda'}(x_1, \ldots, x_n)$$
where λ' is the conjugate of λ.

(b) Give a bijection that goes directly between vertex configuration in Z_λ and those in $\tilde{Z}_{\lambda'}$ (that is, do not rely on SSYT as an intermediary for example).

Bonus Problem 5: DWBC and Schur polynomials
In this problem, we return to DWBC again. Recall that the number of alternating sign matrices of size N is given by

$$\prod_{i=0}^{N-1} \frac{(3i+1)!}{(n+i)!} \tag{1}$$

but, as we saw in lecture and the previous problem sheet, it is also equal to $(q - q^{-1})^{N^2}$ times the partition function of the DWBC six vertex model

$$Z_N(z_1, \ldots, z_N; w_1, \ldots, w_N) =$$
$$(-1)^{N(N-1)/2} \frac{\prod_{i,j=1}^{N} a(z_j/w_i) b(z_j/w_i)}{\prod_{1 \le i < j \le N} b(z_i/z_j) b(w_i/w_j)} \det \left(\frac{c(z_j/w_i)}{a(z_j/w_i) b(z_j/w_i)} \right)_{i,j=1,2,\ldots,N}.$$

with $q = e^{i\pi/3}$, $z_i = 1$, and $w_i = q^{-1}$. One way to prove (1) is by carefully computing the determinant as done by Kuperberg. Here we give an alternate proof by showing that the partition function is in fact a Schur polynomial.

Let $\lambda(N)$ be the double staircase partition
$$\lambda(N) = (N-1, N-1, N-2, N-2, \ldots, 1, 1, 0, 0).$$

First, we will show that $s_{\lambda(N)}$ satisfies the recursion relation
$$s_{\lambda(N)}(x_1, \ldots, x_j = q^2 x_i, \ldots, x_{2N}) = \left(\prod_{\substack{k=1 \\ k \ne i,j}}^{2N} (x_k - q^{-2} x_i) \right) s_{\lambda(N-1)}(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{2N})$$

for any $i < j$, where we skip the variables marked with a hat. By symmetry it is enough to prove this for $i = 1$, $j = 2$.

(a) Show that $s_{\lambda(N)}(x_1, \ldots, x_{2N})$ has degree at most $N-1$ in each variable.

(b) Suppose we set $x_2 = q^2 x_1$. By writing the Schur polynomial as
$$s_\lambda(x_1, \ldots, x_{2N}) = \frac{\det\left(x_j^{\lambda_i + 2N - i}\right)}{\prod_{1 \le i < j \le 2N} (x_j - x_k)}$$
show $s_{\lambda(N)}(x_1, q^2 x_1, x_3, \ldots, x_{2N})$ is zero whenever $x_k = q^{-2} x_1$.

(Hint: write out the first, second, and k^{th} columns of the matrix in the numerator. Show that, since $q = e^{i\pi/3}$, with the given substitution the three columns are linearly dependent, so the numerator is zero. Note that the values of x_1, x_2, x_3 are still distinct so the denominator is non-zero.)

(c) The previous part shows that we have

$$s_{\lambda(N)}(x_1, q^2x_1, x_3, \ldots, x_{2N}) = \left(\prod_{\substack{k=1 \\ k\neq i,j}}^{2N} (x_k - q^{-2}x_i) \right) P(x_1, \ldots, x_{2N})$$

where P is a polynomial. By a degree counting argument, show that P cannot depend on x_1 or x_2.

(d) Finally, by setting $x_1 = 0$ in $s_{\lambda(N)}(x_1, q^2x_1, x_3, \ldots, x_{2N})$, show

$$P(x_1, \ldots, x_{2N}) = s_{\lambda(N-1)}(x_3, \ldots, x_{2N}).$$

Now recall that the recursive formula for the DWBC partition function is given by

$$Z_N(z_1, \ldots, z_N; w_1 = z_1, w_2, \ldots, w_N) =$$
$$c(z_1/w_1) \left(\prod_{i=2}^{N} a(z_1/w_i)a(z_i/w_1) \right) Z_{N-1}(z_2, \ldots, z_N; w_2, \ldots, w_N).$$

(e) By comparing the two recursion relations show that

$$Z_N(z_1, \ldots, z_N; w_1, \ldots, w_N)\big|_{q=e^{i\pi/3}} =$$
$$(-1)^{N(N-1)/2}(q - q^{-1})^N \prod_{i=1}^{N} (qz_iw_i)^{-(N-1)} s_{\lambda(N)}(z_1^2, \ldots, z_N^2, q^2w_1^2, \ldots, q^2w_N^2).$$

Specifically, using the recursion relation for the Schur polynomial, show that the expression on the RHS of the above satisfies the recursion relation for the DWBC six vertex model.

(Note that when $q = e^{i\pi/3}$ we see that the DWBC partition function is symmetric in the full set of variables $\{z_i\} \cup \{qw_i\}$ rather than just each set individually.)

(f) By setting $z_i = 1$ and $w_i = q^{-1}$ show that the number of ASMs of size N is given by

$$(q - q^{-1})^{-N(N-1)} s_{\lambda(N)}(\underbrace{1, \ldots, 1}_{2N}) = \prod_{i=0}^{N-1} \frac{(3i+1)!}{(n+i)!}.$$

(Hint: the identity

$$s_\lambda(\underbrace{1, \ldots, 1}_{2N}) = \prod_{1\leq i<j\leq 2N} \frac{\lambda_i - i - \lambda_j + j}{j - i}$$

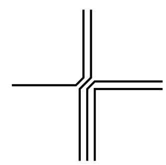
will be useful.)

Lecture B7: A Colored Vertex Model and Littlewood-Richardson Coefficients

Types of Generalizations:
We start this lecture discussing ways to generalize the types of vertex models we have seen so far. So far our vertex model has consisted of one color of path with only one path allowed per edge. To help understand the generalization, we need a bit about the underlying representation theory.

The R-matrix for the six vertex model can be constructed from the quantum group $U_q(\hat{\mathfrak{sl}}_2)$. In fact, let V be the standard represented (the 2-dimensional irreducible representation) then our R-matrix can be seen as a map $R : V \otimes V \to V \otimes V$.

One way to generalize this is to let V be a higher dimensional irreducible representation. In this case we get vertices that can look like



in which we now allow multiple paths per edge.

Another type of generalization is to increase the rank. That is, we consider $U_q(\hat{\mathfrak{sl}}_3)$ or more generally $U_q(\hat{\mathfrak{sl}}_n)$. In this case, we get vertex models with two colors of paths (or in general $n-1$ colors). We'll consider one such vertex model today.

Littlewood-Richardson Coefficients:
Recall that the Schur polynomials for a basis for the symmetric polynomials. Since the product of two symmetric polynomials is symmetric, we can expand the product of two Schur polynomials in the Schur basis. We have

$$s_\lambda(x_1,\dots,x_k)s_\mu(x_1,\dots,x_k) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(x_1,\dots,x_k)$$

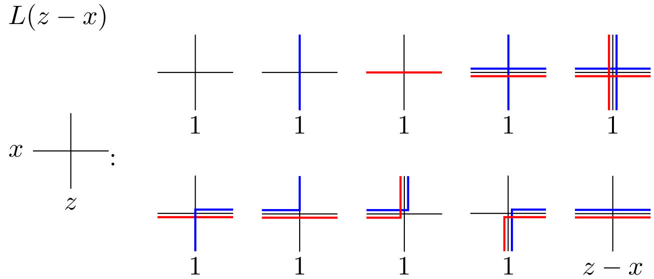
where the structure constants $c^\nu_{\lambda\mu}$ are known as *Littlewood-Richardson coefficients.*

It turns out these coefficients are always non-negative integers. There are several combinatorial formulas for these coefficients. We would like to give a vertex model formula for them. The basic idea is very similar to that of the Cauchy identity:

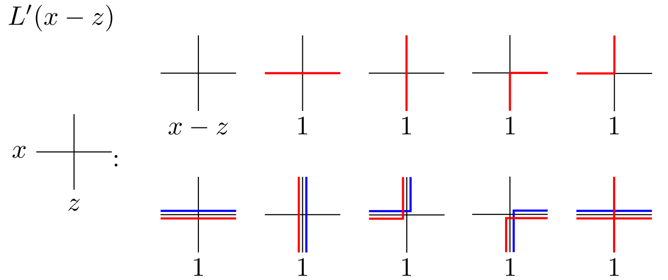
$$\underbrace{s_\lambda(x_1,\dots,x_k)s_\mu(x_1,\dots,x_k)}_{\text{partition func. of some vertex model}} \underbrace{=}_{\text{many applications of YBE}} \underbrace{\sum_\nu c^\nu_{\lambda\mu} s_\nu(x_1,\dots,x_k)}_{\text{partition func. of another vertex model}}$$

where the sum on the RHS comes from looking at where the paths cross some slice. We'll then find the part of the RHS that contribute the Littlewood-Richardson coefficients. Now let us make these ideas precise.

A Colored Vertex Model:
Just as for the Cauchy identity, we will need two types of vertices L and L'. The vertices and their weights are given by



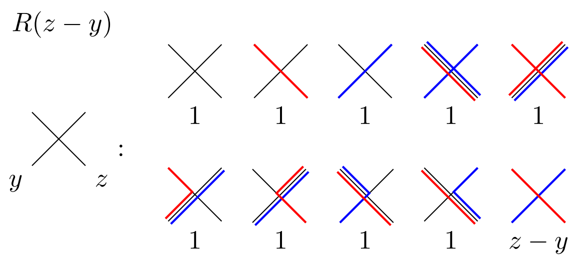
and



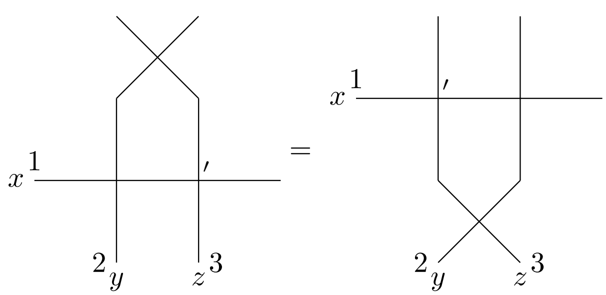
- Note that we no longer draw arrows and that it should be clear from context when an edge is unoccupied versus undetermined.
- Both vertex models consist of paths traveling up and right. Now there are two colors of path. The two colors, however, are not independent and the allowed behavior of one color of path is restricted by the behavior of the other.
- The weights come with a column parameter z that eventually we will set to zero (although everything we do works fine with $z \neq 0$).

I want to highlight the top row of vertices of L'. These are exactly our five vertices from the lecture on Schur polynomials. In fact, if we set $z = 0$, the weights are the Schur polynomial L' weights without the unnecessary factors from paths exiting at the top. We will use this to get Schur polynomials out of this vertex model later.

These weights satisfy a YBE with the cross weights are given by



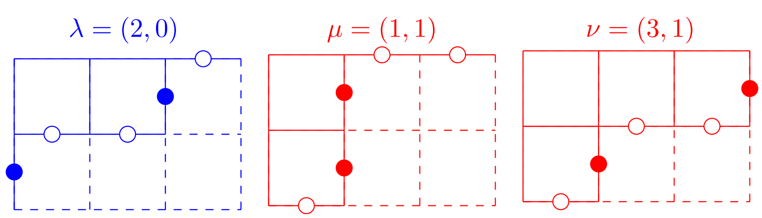
The YBE takes the form



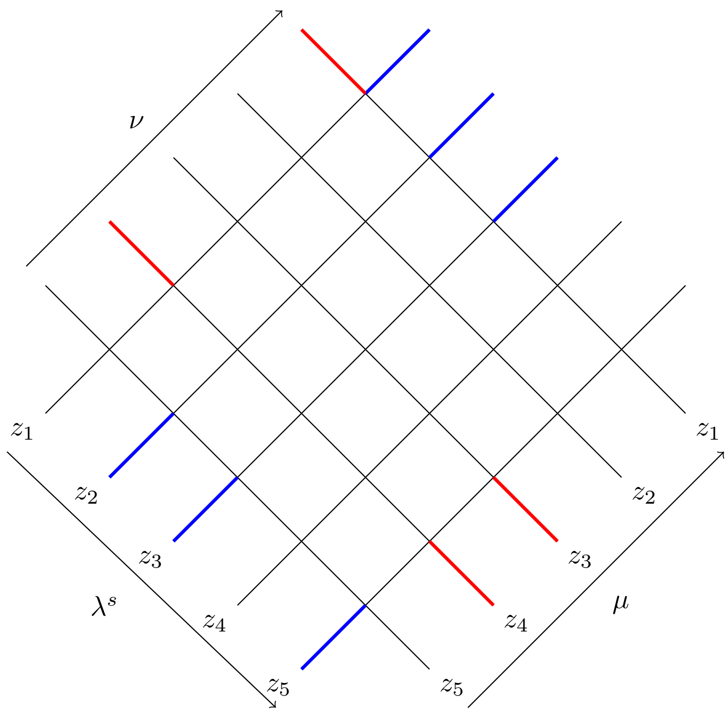
$$L_{12}(y-x)L'_{13}(x-z)R_{23}(z-y) = R_{23}(z-y)L'_{13}(x-z)L_{12}(y-x)$$

where note the cross gets pushed vertically rather than horizontally like we have seen in the previous lectures.

Now let's get to the punchline of the lecture. Fix $n, k \in \mathbb{Z}_{>0}$. Consider partitions λ, μ, and ν each of which fit in an $k \times (n-k)$ box. For example, let $k = 2$ and $n = 5$. One possible choice of partitions is



We draw the corresponding part of the Maya diagram as well (but now don't bother re-orienting the diagrams). Now consider an $n \times n$ square



- There are k red paths and $n-k$ blue paths.
- Three of the boundary conditions are determined by the Maya diagrams of the partitions with the arrows indicated the order to place the paths. The NE boundary has the blue paths fully packed to the left. All uncolored boundary edges are empty.
- By λ^s we mean the Maya diagram for λ with holes and particles swapped.
- We use the same $z_1, \ldots, z_n$ on the SE-NW strands and SW-NE strands.

Let $Z^{\nu}_{\lambda\mu}(z_1, \ldots, z_n)$ be the partition function.

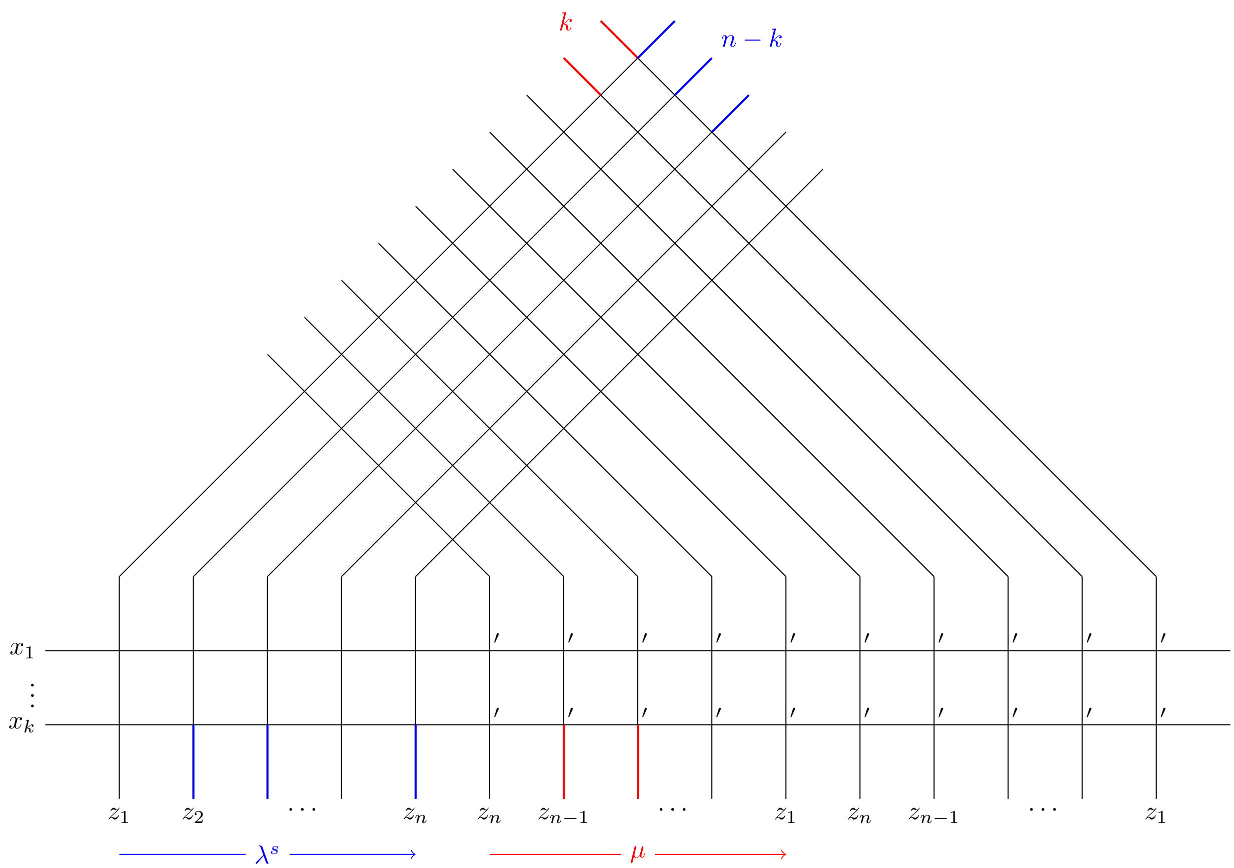
Theorem 0.20.

$$Z^{\nu}_{\lambda\mu}(0, \ldots, 0) = c^{\nu}_{\lambda\mu}$$

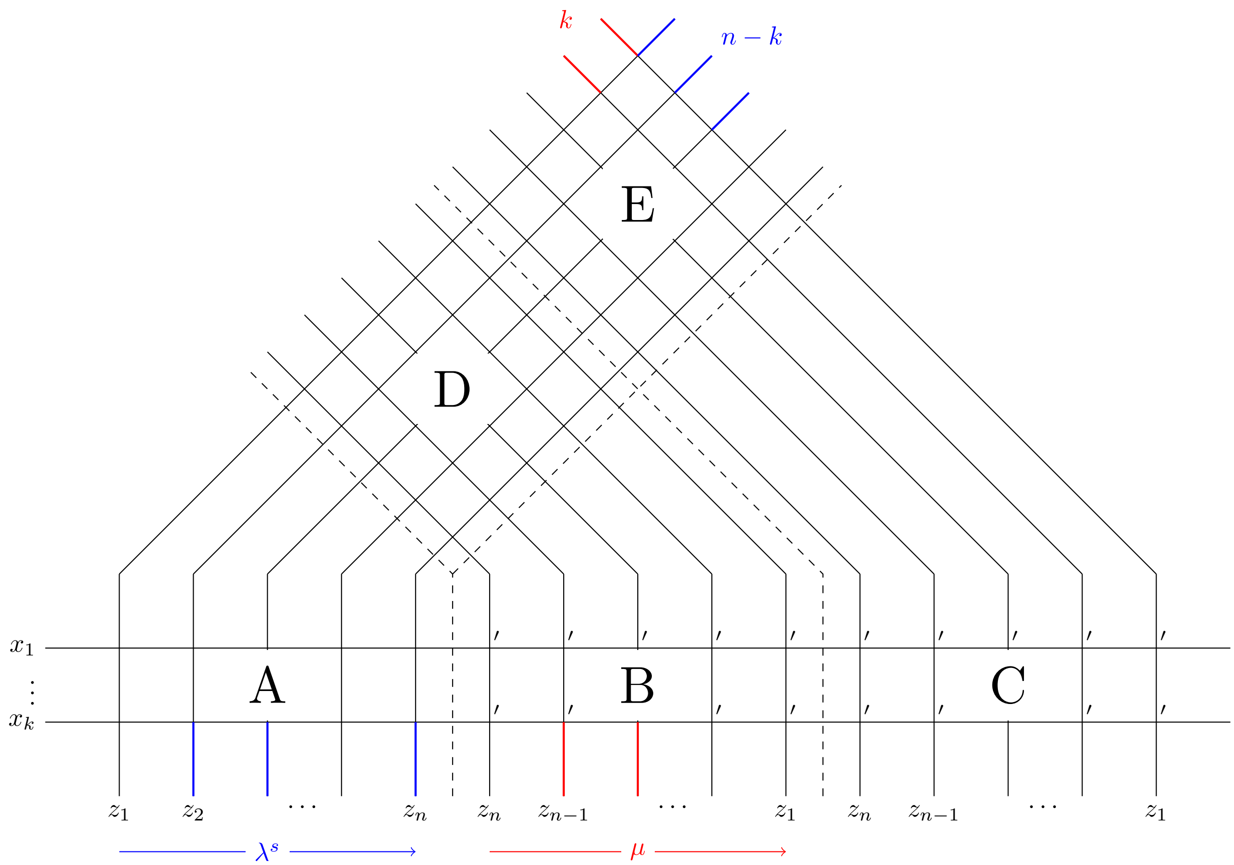
This gives a combinatorial formula for the Littlewood-Richardson coefficients. We can count the possible vertex configurations and that will give us $c^{\nu}_{\lambda\mu}$.

The Set-up:
Now we will sketch a proof of the above theorem. We'll assume that $\lambda_1 + \mu_1 + k \leq n$. This is always possible by choosing n sufficiently large. (It is possible, and one should try, to prove that increasing n does not

change the partition function $Z^{\nu}_{\lambda\mu}$.) First, we introduce the first domain we need:



with all other boundary edges empty. To understand the partition function, we will break the domain into pieces as follows



and work through the pieces.

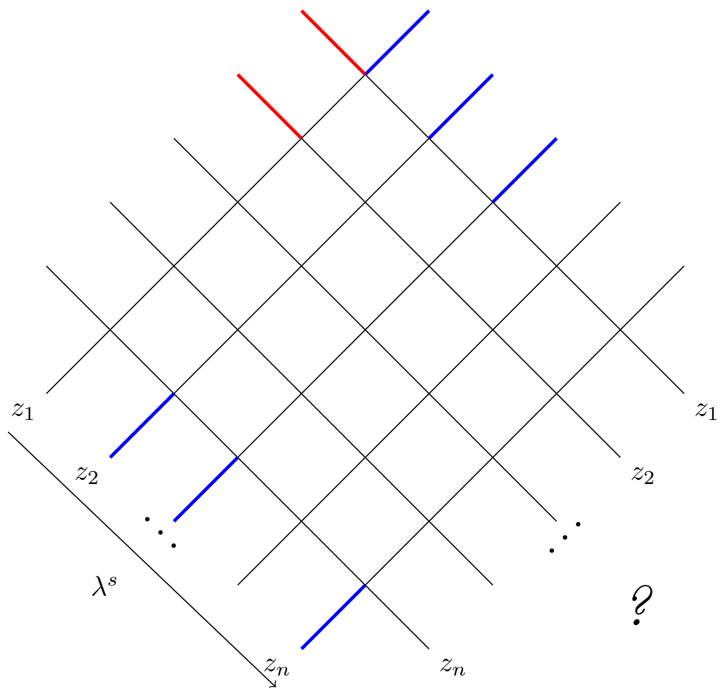
A: There are only blue paths. Since we are using the L weights here, the paths are forced to go straight vertically.

B and D: We will argue that the red paths in B must exit B to the right. Suppose a red path exited B at the top and enters D. Note that it must exit a vertex traveling NE at least $n-k+1$ times. Examining the R weights, we see that can only exit a cross vertex traveling NE by turning right after entering from the SE. It then must turn left and leave the blue path. Thus every time a red path exits a vertex traveling NE it must cross a blue path. Since there are only $n-k$ blue paths to cross, the red path will not be able to make it to its endpoint.

As there are then no red paths in D, by examining the R weights, we see that the blue paths must travel straight NE.

E: From the previous point, we know the blue paths enter the SW boundary of E according to λ^s. It turns out that with these boundary conditions for E is a unique configuration in this region. We make this precise in the following lemma.

Lemma 0.21. *fix λ that fits in a $k \times (n-k)$ box. Consider an $n \times n$ square of cross vertices with boundary conditions on the NE, NW, and SW given by*

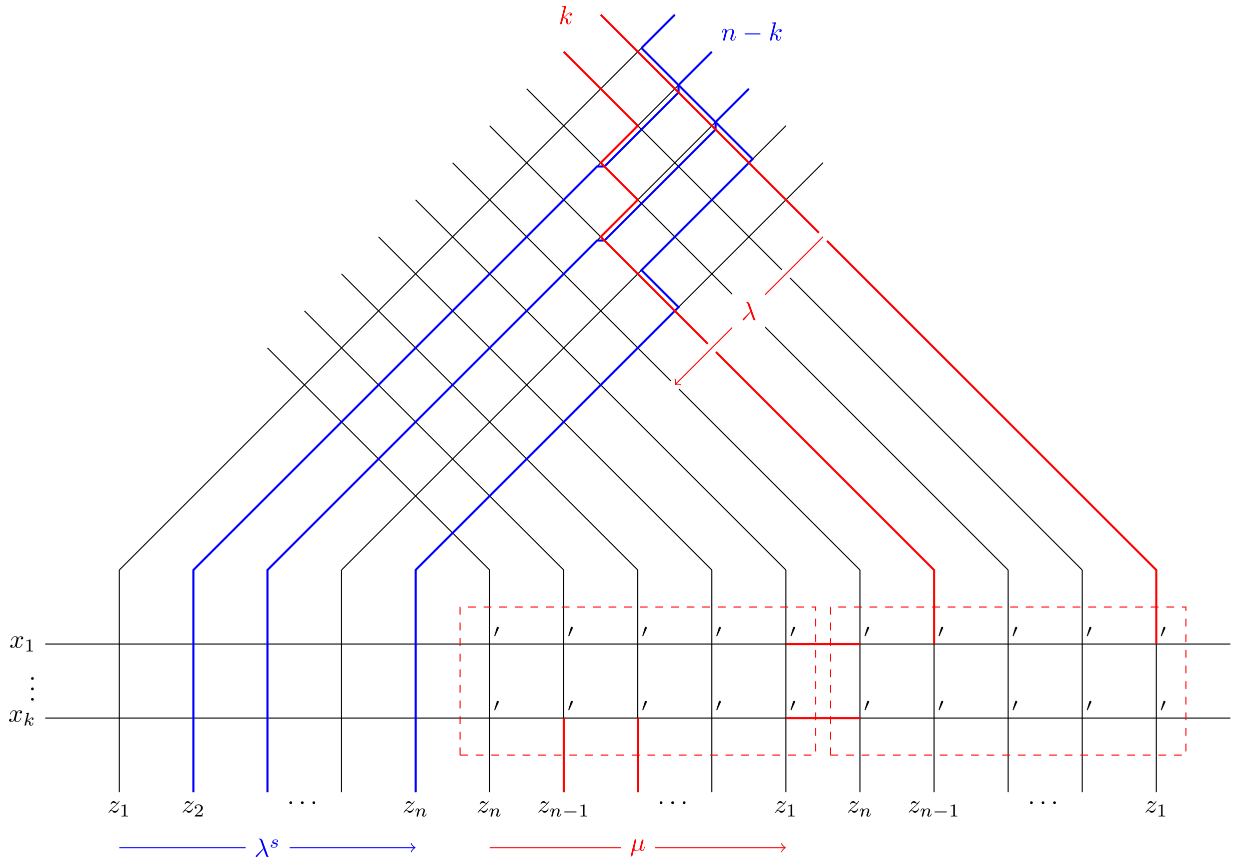


Then there is a valid configuration iff the SE boundary is given by the Maya diagram for λ placed from right to left.

We leave the proof of this lemma to the exercises.

C: The previous analysis tells us that we have a red path entering from every row on the left of C and they exit at the top at positions given by λ.

Filling this all in, we have

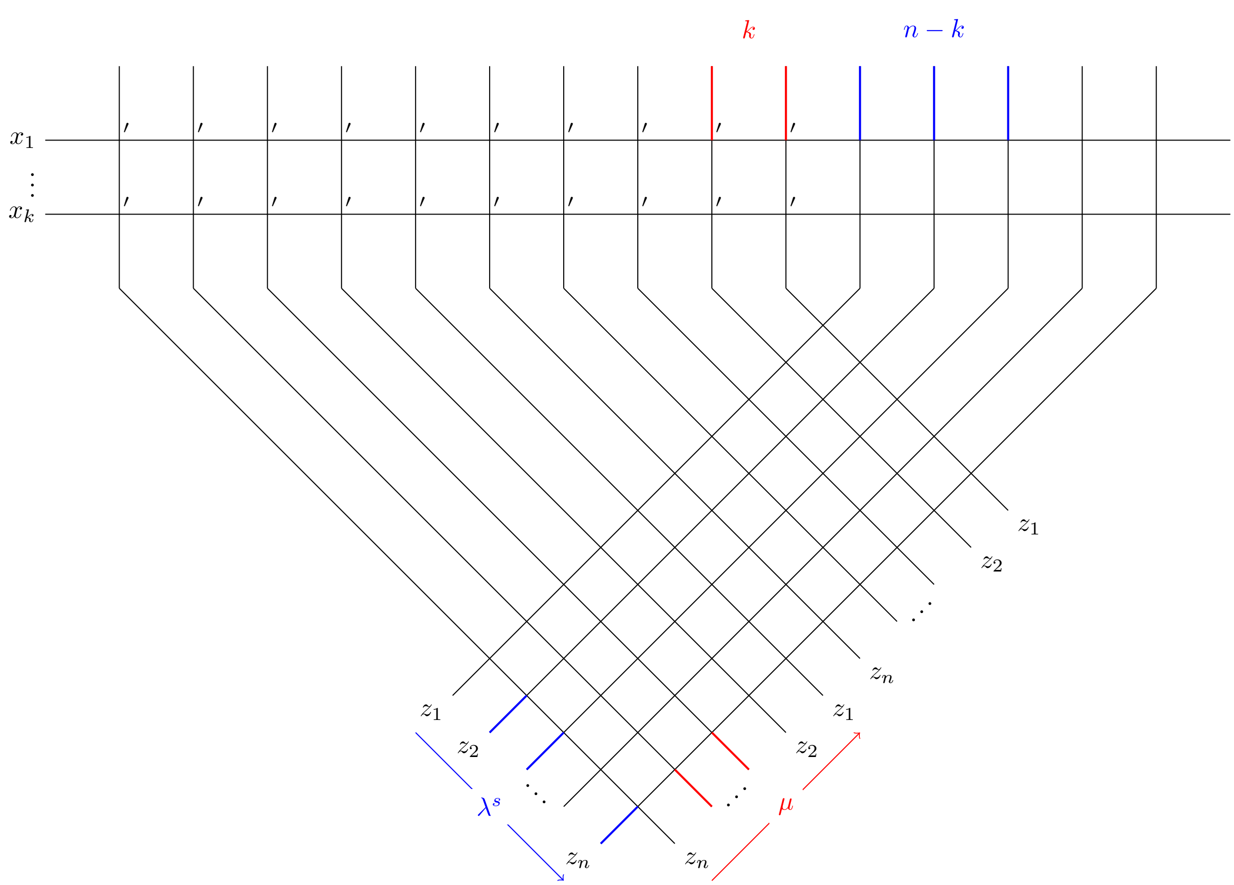


where the only freedom is inside the dashed regions. All together we see that the partition function is equal to

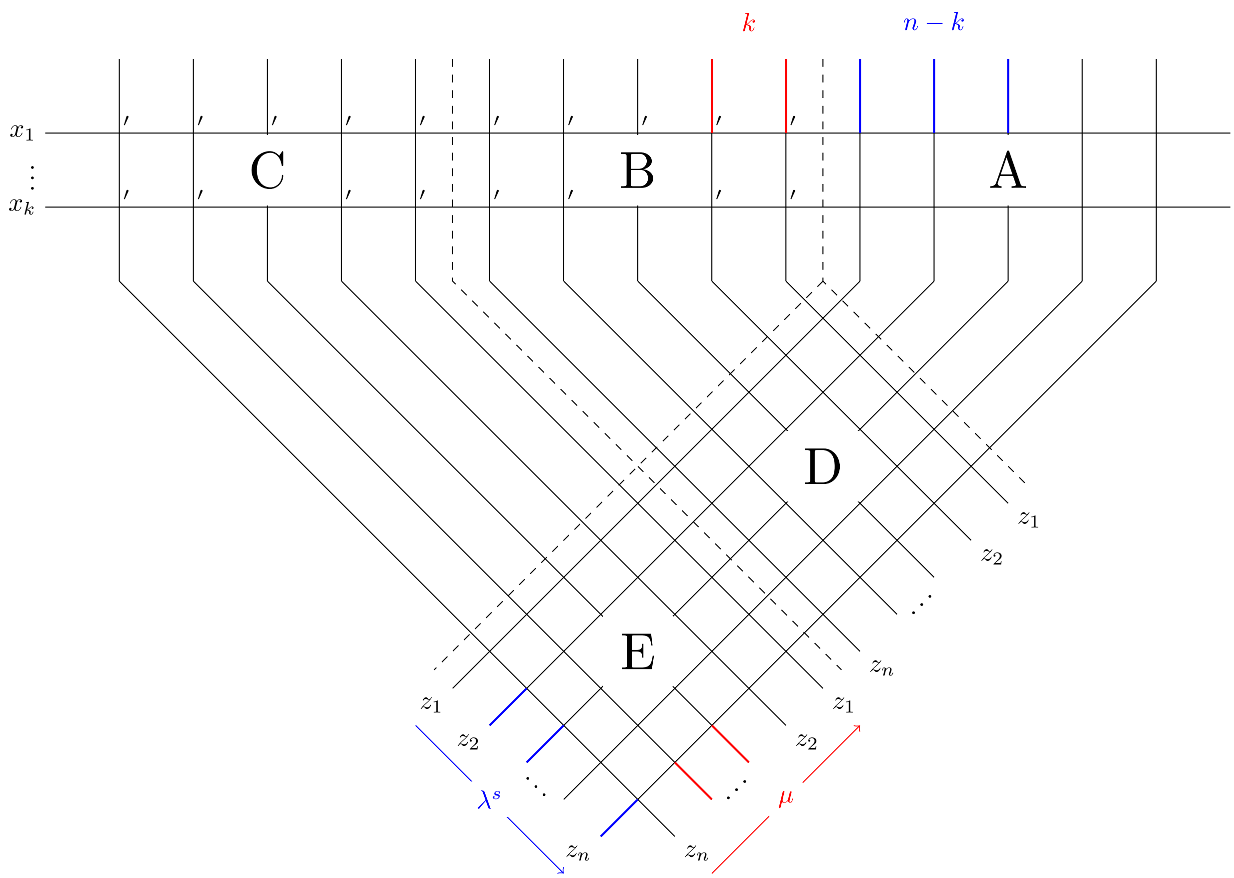
$$s_\lambda(x_1, \ldots, x_k)s_\mu(x_1, \ldots, x_k)$$

when $z_1 = \cdots = z_n = 0$, with s_λ coming from C (noting the invariance under 180 degree rotation of the weights) and s_μ coming from B. (Every other piece has no freedom and has weight 1.)

Repeatedly using the YBE to push the crosses down gives

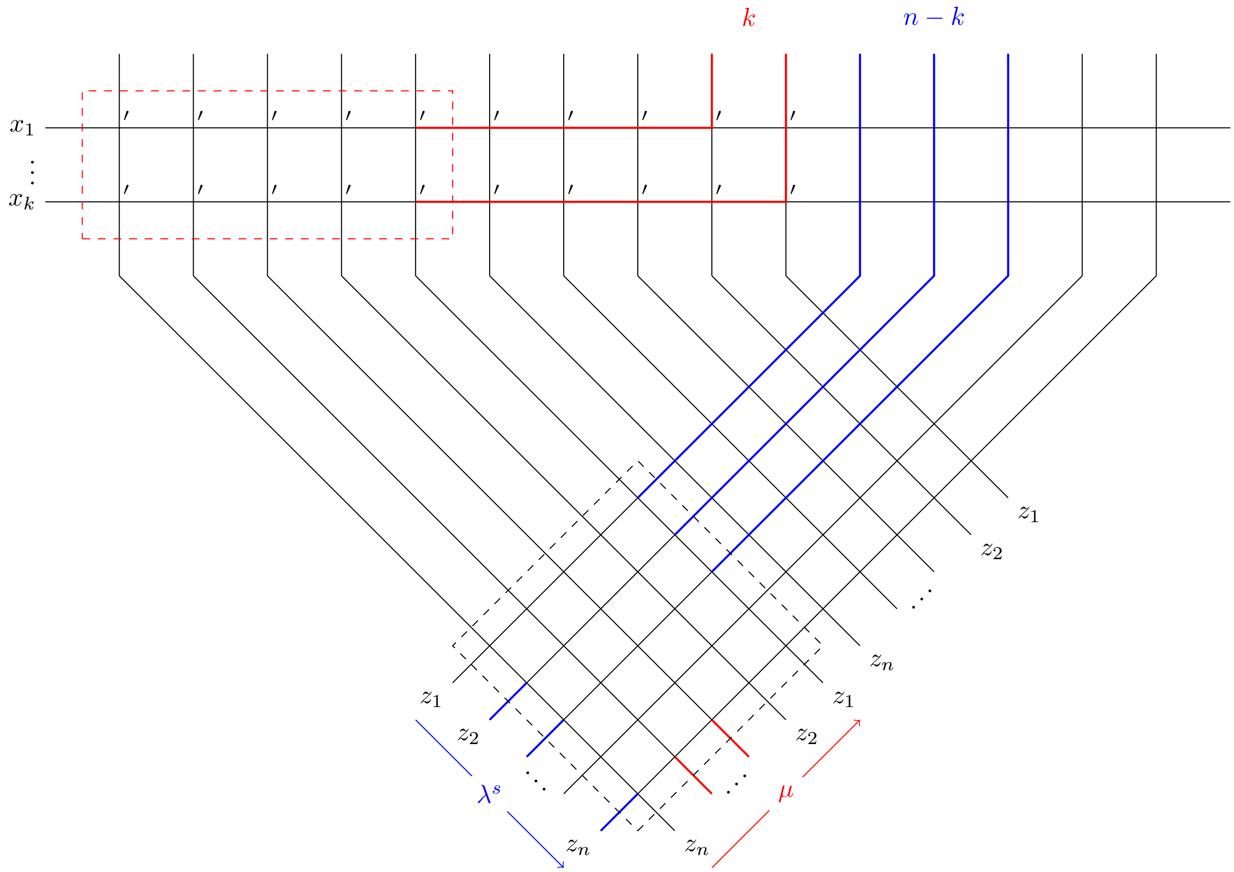


Again we will break this into pieces and work through each piece.



A: This region can only contain blue paths. Checking the L weights, we see the paths must be straight vertical.
E: We want to argue that the red paths here must exit at the NW boundary. Note from the R weights, a red path can only shift to the NE when it crosses a blue path. Similarly, a blue path can only shift to the NW when it crosses a red path. At each crossing only one color can make a shift. Note that any blue paths corresponding to the top of the $k \times (n-k)$ box (like the bottom-right most blue path in our example) must eventually shift k times to reach its end point, exhausting the supply of red paths. Thus there are only λ_1 blue paths at which a red path can shift. Since the rightmost red path starts $\mu_i + k$ strands from the bottom corner, it can shift as far as $\lambda_1 + \mu_i + k$ strands from the bottom corner. Recall that we assumed $\lambda_1 + \mu_i + k \leq n$. This ensures that the red paths cannot shift enough to leave the region E to the NE. We see that they must exit to the NW.
D: By the previous point there are no red paths in D, only blue. The weights ensure each must travel in a straight line.
C: The k red paths from E enter here at some positions and the boundary conditions force that the exit the region to the right. Note that the occupy every exiting row.
B: With a red path entering at every row on the left and exiting at the top fully packed to the right, there is only one possible configuration.

Putting this together the picture looks like



where the only freedom is in the dashed regions. Overall we have that the partition function is

$$\sum_{\nu} Z^{\nu}_{\lambda\mu}(0,\ldots,0)s_{\nu}(x_1,\ldots,x_k)$$

when $z_1 = \cdots = z_n = 0$, where the sum over ν is coming from looking at where the paths exit E to the NW (or equivalently, where they enter C from the bottom). Equating this with what we had before moving all the crosses gives

$$s_{\lambda}(x_1,\ldots,x_k)s_{\mu}(x_1,\ldots,x_k) = \sum_{\nu} Z^{\nu}_{\lambda\mu}(0,\ldots,0)s_{\nu}(x_1,\ldots,x_k).$$

Comparing this to

$$s_{\lambda}(x_1,\ldots,x_k)s_{\mu}(x_1,\ldots,x_k) = \sum_{\nu} c^{\nu}_{\lambda\mu}s_{\nu}(x_1,\ldots,x_k).$$

which defines the Littlewood-Richardson coefficients, we see that

$$Z^{\nu}_{\lambda\mu}(0,\ldots,0) = c^{\nu}_{\lambda\mu}$$

as desired.

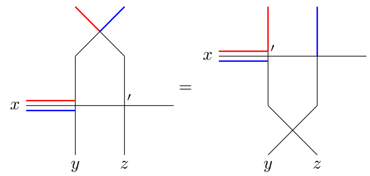
Note: While we have focused on the case with $z_1 = \cdots = z_n = 0$ everything makes sense with the $z_i \neq 0$. With $z_i \neq 0$, instead of Schur polynomials we have what are known as factorial Schur polynomials, and the quantities

$$Z^{\nu}_{\lambda\mu}(z_1,\ldots,z_n)$$

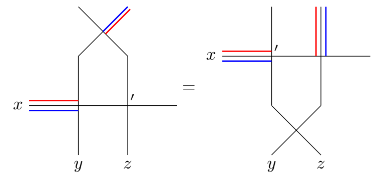
are the coefficient needed for expanding the product of two factorial Schur polynomials into a linear combination of factorial Schur polynomials.

Lecture B7 Exercises.

Problem 1: YBE
Using the L, L', and R weights from lecture, check the Yang-Baxter equation for the boundary conditions

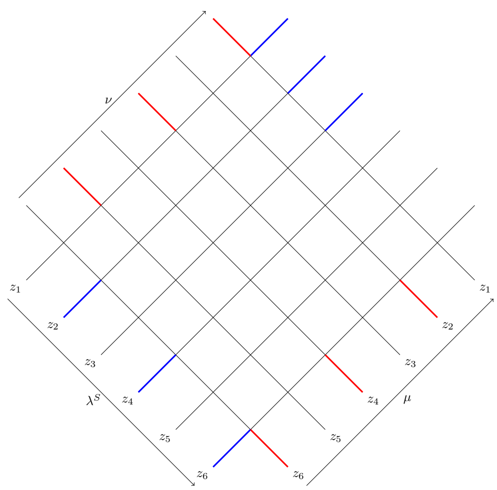


and



where all other boundary edges are empty.

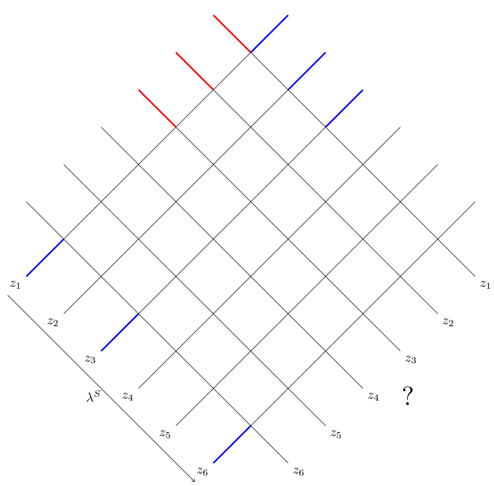
Problem 2: Littlewood-Richardson Coefficients via the vertex model
Let $\lambda = (2,1,0)$, $\mu = (2,1,0)$ and $\nu = (3,2,1)$. Using the two-color R weights from lecture, compute the partition function of



with $z_1 = \ldots = z_6 = 0$, where all other boundary edges are empty. Deduce that $c^{\nu}_{\lambda\mu} = 2$.

Problem 3: Littlewood-Richardson Coefficient Lemma

(a) Let $\lambda = (2,2,1)$, $n = 6$, and $k = 3$. Using the two-color R weights from lecture show that



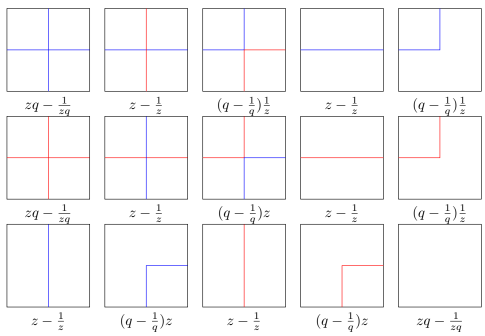
only has valid configuration if the SE boundary is given by the Maya diagram for λ in an $k \times (n-k)$ box, placed going from the East corner to the South corner.

(b) **Optional:** Prove that this holds generally.

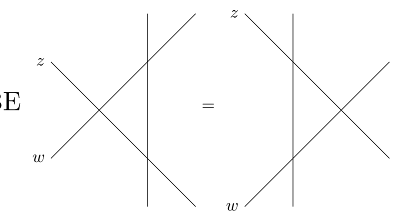
Problem 4: Littlewood-Richardson Coefficients for factorial Schur polynomials
Let $c^{\nu}_{\lambda\mu}(z_1, \ldots, z_n)$ be the partition function for our vertex model that computes the Littlewood-Richardson coefficients. Fix $n = 4$, $k = 2$, and consider $\lambda = (2, 0)$, $\mu = (2, 0)$, and $\nu = (2, 1)$. Note that $c^{\nu}_{\lambda\mu}(0, 0, 0, 0) = c^{\nu}_{\lambda\mu} = 0$ since $|\lambda| + |\mu| > |\nu|$. Show that

$$c^{\nu}_{\lambda\mu}(z_1, z_2, z_3, z_4) = z_4 - z_2.$$

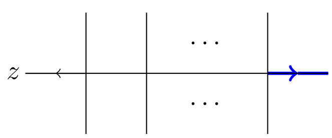
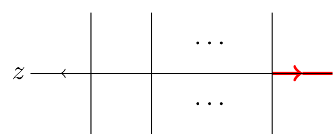
Problem 5: Jimbo's two color vertex model
Consider the two-colored weights



with YBE



where the cross vertex uses parameter z/w. Consider the row operators

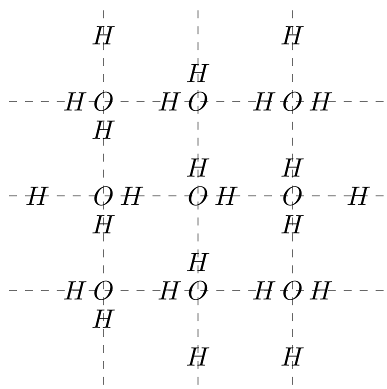
$B_1(z) =$ , $B_2(z) =$ .

Show that they satisfy the relation

$$\left(\frac{zq}{w} - \frac{w}{zq}\right) B_1(w)B_2(z) = \left(\frac{z}{w} - \frac{w}{z}\right) B_2(z)B_1(w) + \left(q - \frac{1}{q}\right) \frac{w}{z} B_1(z)B_2(w).$$

Lecture B9: Limit Shapes and Large Tilings

Recall:
Originally, the six vertex model was introduced by Linus Pauling to study 2d ice:

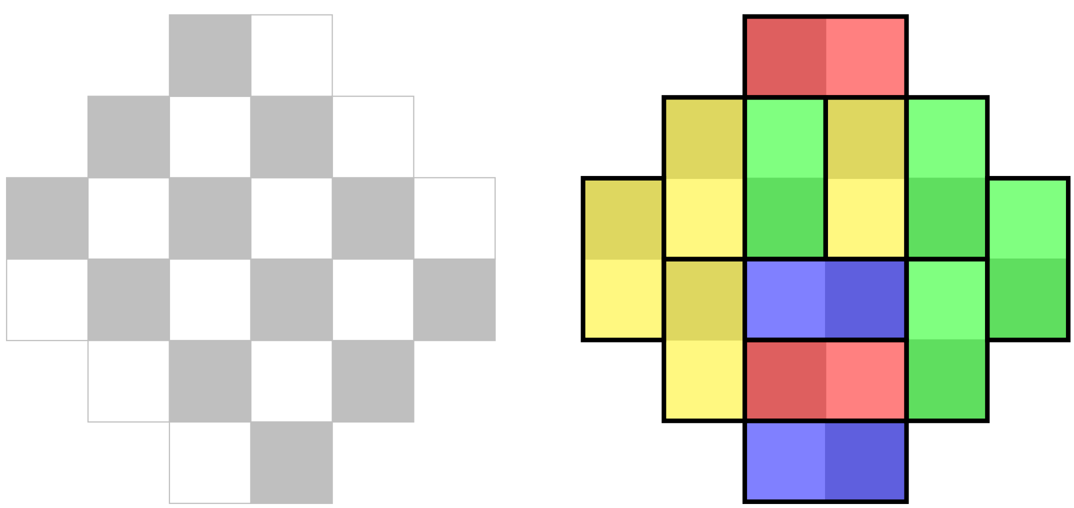


He was interested in periodic b.c. on a $N \times N$ square, with uniform weights. Pauling conjectured

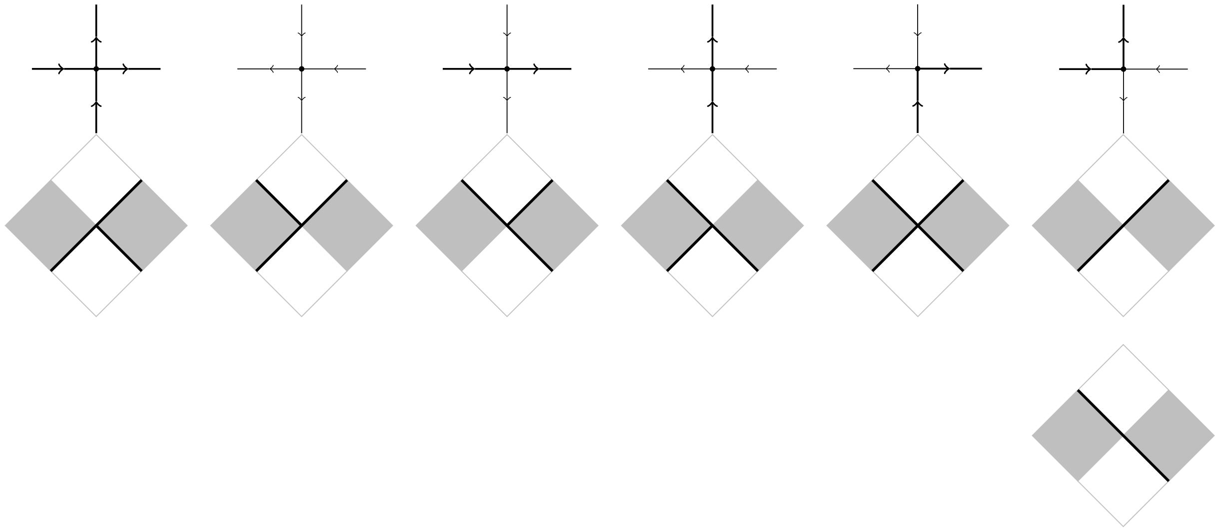
$$Z_N \sim e^{N^2 c} \text{ with } c \approx 1.5$$

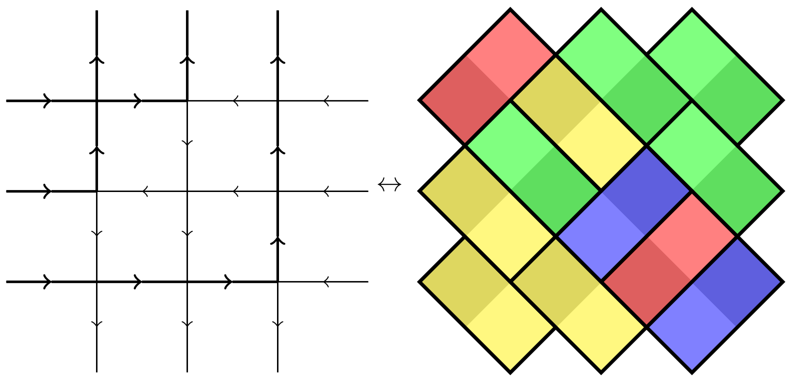
We see that from the very beginning there was interest in asymptotics, that is, the behavior of the model as the size of the domain goes to infinity. (In this case, Pauling was interested in the asymptotic growth rate.)

Rather than talk about the six vertex model, we will focus on domino tilings of the Aztec diamond.



Above is the Aztec diamond of rank $m = 3$ and one possible domino tiling. (The colors of the dominos are used to distinguish the four possible orientations.) Recall that there is a mapping between DWBC 6V (with weights $a = 1, b = 1, c = \sqrt{2}$) and domino tilings of the AD:



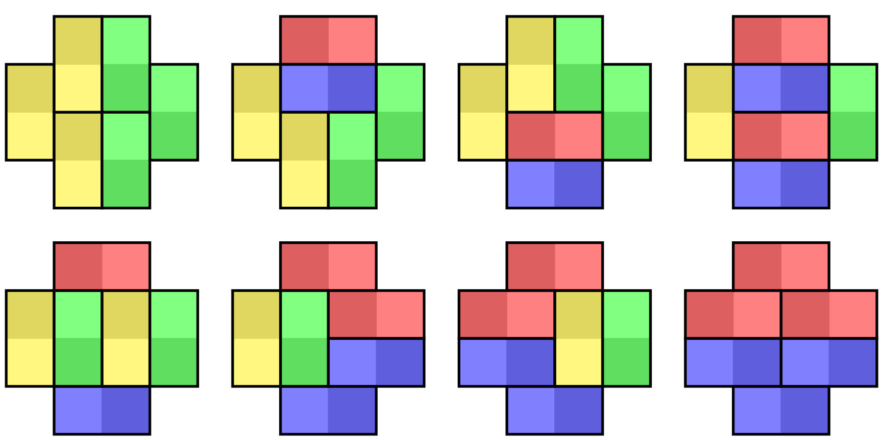


Large Tilings:
A first question one could ask is how the number of tilings grows as the rank N increases.

Theorem 0.22 (Elkies-Kuperberg-Larsen-Propp (1992))**.** *The number of tilings of the Aztec diamond of rank N is given by*

$$2^{\binom{N+1}{2}} = 2^{N(N+1)/2}.$$

The 8 tilings of rank 2:



Before we get to more probabilistic inquiries let us do a brief warm up.

Law of large numbers: Flip a fair coin N times. Let $X_i = 1$ if i^{th} flip heads or 0 otherwise.

Q: What proportion of the coin flips are heads?

$$\text{Number of heads: } S_N = X_1 + \ldots + X_N$$

$$\mathbb{P}(|S_N/N - 1/2| < \epsilon) \to 1 \text{ as } N \to \infty.$$

- As a parameter governing the size gets very large the distribution concentrates at a particular value.

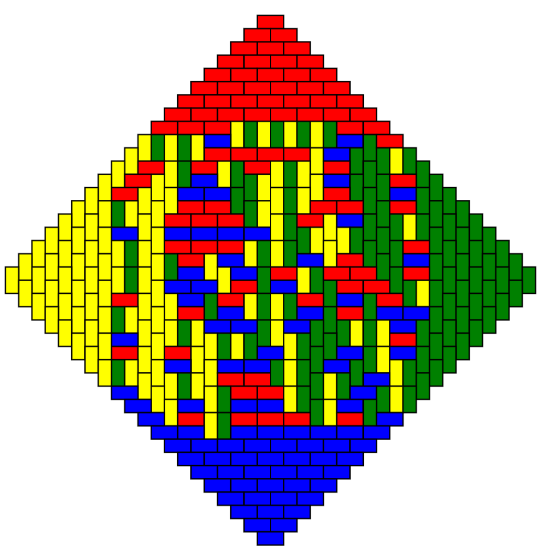
Central limit theorem: The proportion will not always be exactly $\frac{1}{2}$, there will be some error.

Q: What is the distribution of the error?

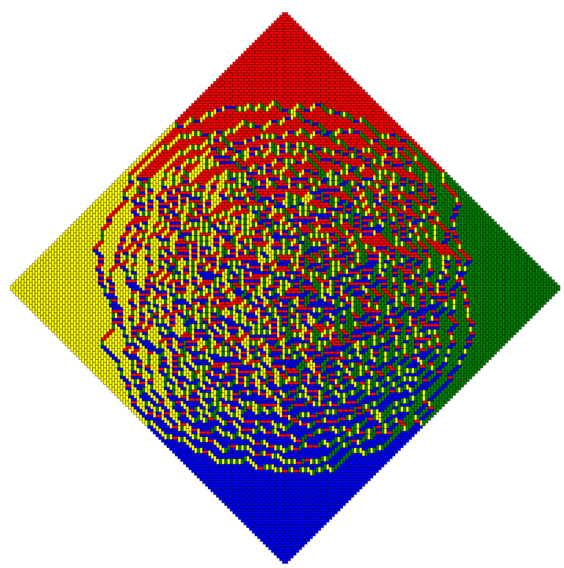
$$(S_N - N/2)/\sqrt{N} \xrightarrow{d} \mathcal{N}(0, 1/4)$$

- After appropriate rescaling ($\sqrt{N}$) the error converges in distribution to normal random variable.
- **Universality:** This is true in much more generality then just coin flips. (For example, any sequence X_i of i.i.d. random variables with finite mean and variance.)

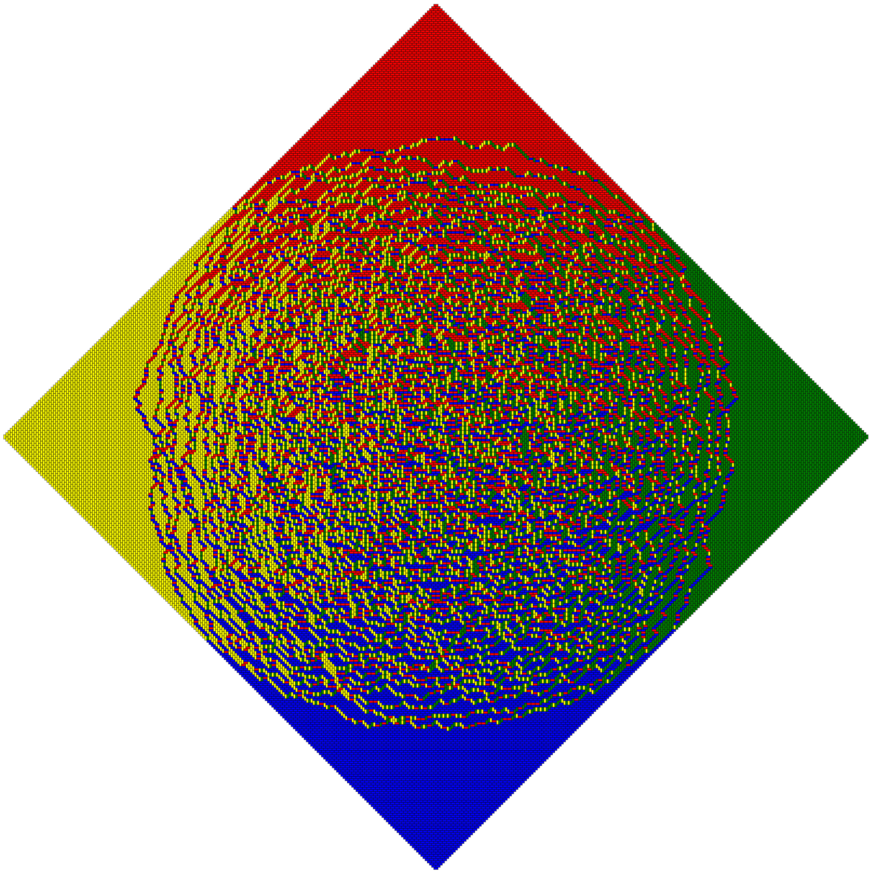
Now let us look at some large random tilings. The below tilings were chosen uniformly at random from all possible tilings of the given rank.



Rank 20



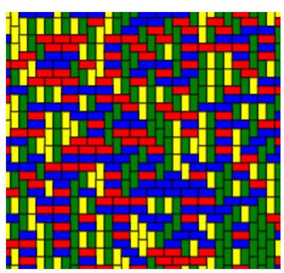
Rank 100

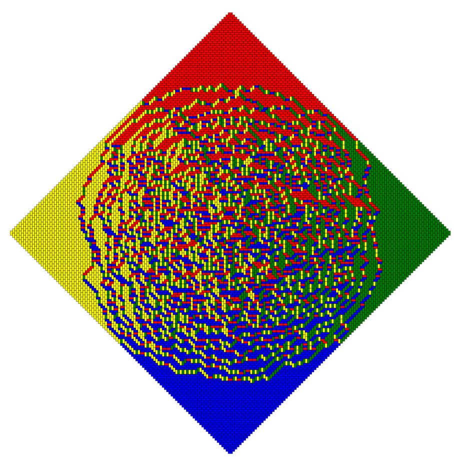


Rank 200

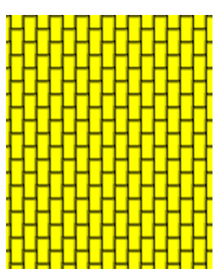
We see some interesting geometric features appearing.

Disordered center:





Brickwork corners:



- Brickwork corners (known as the frozen regions) are not forced by the boundary conditions. It is a type of law of large numbers known as the **limit shape phenomenon.**
- Much effort has been put into understanding this phenomenon.

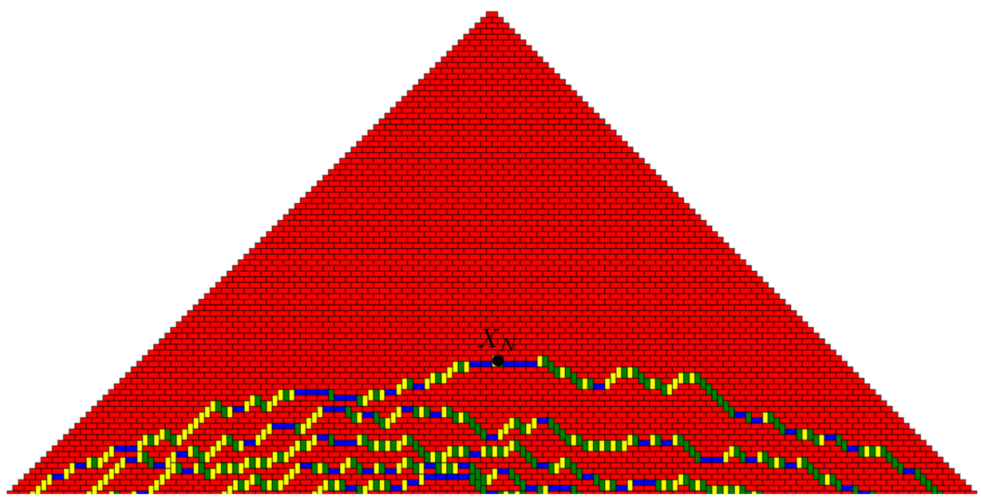
As a first example, let us consider the boundary between the disordered region and the frozen region. From the simulations, the boundary seems to approach a circle. This can be made precise.

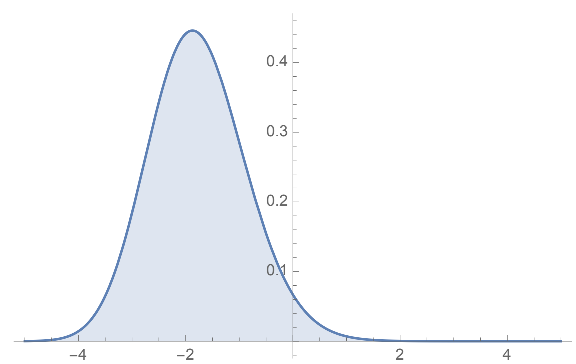
Theorem 0.23 (Jockush-Propp-Schor (1995))**.** *For all $\epsilon > 0$, there exists N such that "almost all" randomly picked domino tilings of rank N or larger have a disordered region whose boundary stays uniformly within distance ϵN from the circle of radius $N/\sqrt{2}$.*

"almost all" = "with probability greater than $1 - \epsilon$."

- As the size parameter gets very large, the distribution concentrates around tilings with this particular geometric feature.
- This is a type of law of large numbers for the tilings.
- The boundary is know as the arctic circle since it separates the inside from the frozen regions.

We can also ask about the error/fluctuation of the actually boundary and the limiting arctic circle. Let X_N be a random variable giving the distance of the boundary from the center of the AD along the vertical axis.





Theorem 0.24 (Johansson (2005))**.**

$$\mathbb{P}\left(\frac{X_N - N/\sqrt{2}}{2^{-5/6}N^{1/3}} \leq s\right) \to \underbrace{F_2(s)}_{\textit{Tracy-Widom cdf}}$$

This is a type of central limit theorem for the tilings. For tilings we have

$$\mathbb{P}\left(\frac{X_N - N/\sqrt{2}}{2^{-5/6}N^{1/3}} \leq \xi\right) \to F_2(\xi)$$

where F_2 is the cdf of the Tracy-Widom distribution. Compare this to CLT for binomial random variables

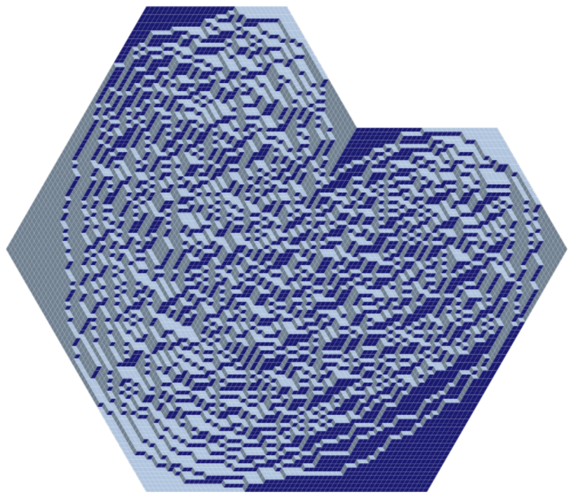
$$\mathbb{P}\left(\frac{S_N - N\mu}{\sqrt{N}\sigma} \leq z\right) \to \Phi(z)$$

where Φ is the standard normal cdf.

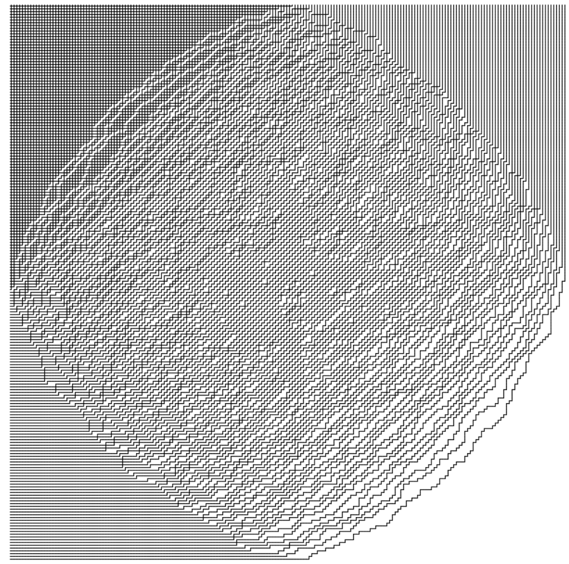
- Very similar flavor! Some differences...
- Scaling by $N^{1/2}$ for the usual CLT, but $N^{1/3}$ for the tiling.
- They have different limiting distribution (Tracy-Widom vs. Gaussian).

(**Note:** There are other types of asymptotic questions on can ask about the tilings, these are just two common examples.)

Many other models exhibit the limit shape phenomenon, arctic curves, Tracy-Widom fluctuations...



Lozenge Tiling



Six Vertex Model Config.

- We say that these phenomenons are universal.
- There are few precise universality results. This is an observed phenomenon.

Another place these fluctuations arise (in fact, we they original arose) is in random matrix theory. Let M_N be an $N \times N$ complex matrix such that

- It is Hermitian, ie $M_{ij} = \overline{M}_{ji}$ and M_{ii} real.
- For $i > j$, the M_{ij} are i.i.d. standard complex normals ($M_{ij} \sim \mathcal{N}(0, 1/2) + i\mathcal{N}(0, 1/2)$).
- The diagonal entries are i.i.d. $M_{ii} \sim \mathcal{N}(0, 1)$.

It's known that the rescaled largest eigenvalue $\lambda_1(N)/\sqrt{N}$ will be

$$\lambda_1(N)/\sqrt{N} \to 2 \quad \text{a.s. when taking } N \to \infty.$$

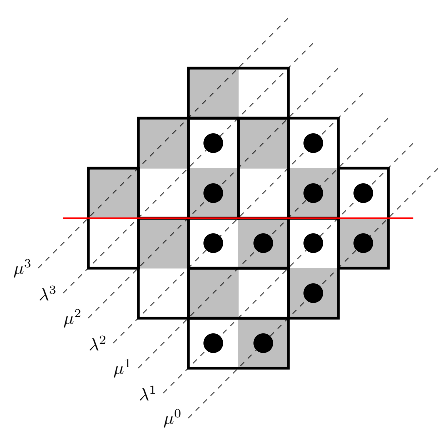
The fluctuations around this limiting value are given by

$$\mathbb{P}\left(N^{1/6}(\lambda_1(N) - 2\sqrt{N}) < \zeta\right) \to F_2(\zeta)$$

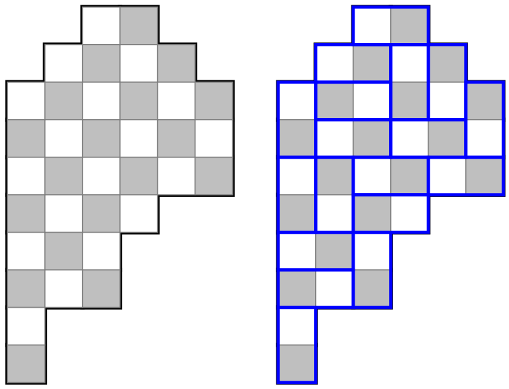
The same Tracy-Widom fluctuations as seen in our tilings!

For the tilings, the underlying algebraic and combinatorial structure is what allows us prove these asymptotic results. In this case, the tilings of the Aztec diamond are intimately related to sequences of interlacing partitions and Schur polynomials, and is an example of a more general construction known as a Schur process.

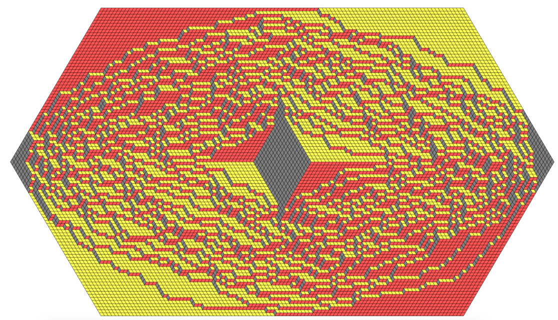
$$\emptyset \preceq' (1,1) \succeq (1,1) \preceq' (2,1) \succeq (1) \preceq' (2) \succeq \emptyset$$



What types of things are people working on:
Rather than just the Aztec diamond people study more complicated domains.



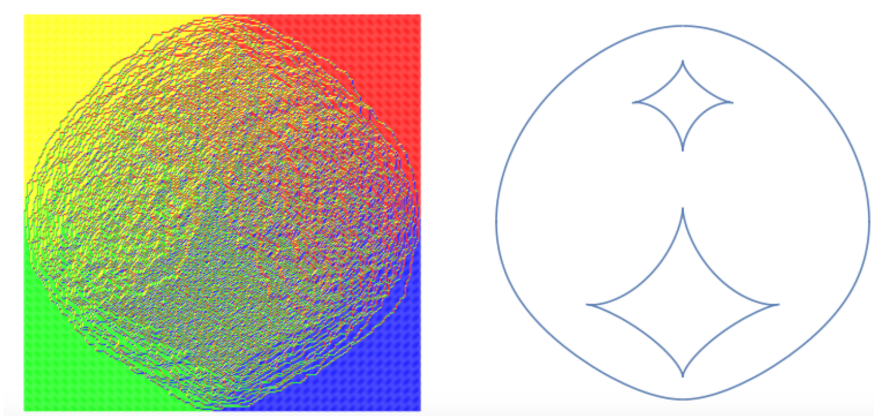
Aztec Triangle
Corteel-Huang-Kratenthaler (2023)



Holey Hexagon (note the hole in the center)
simulation courtesy of Leonid Petrov

In this case, there are often interesting questions about enumeration and about asymptotics.

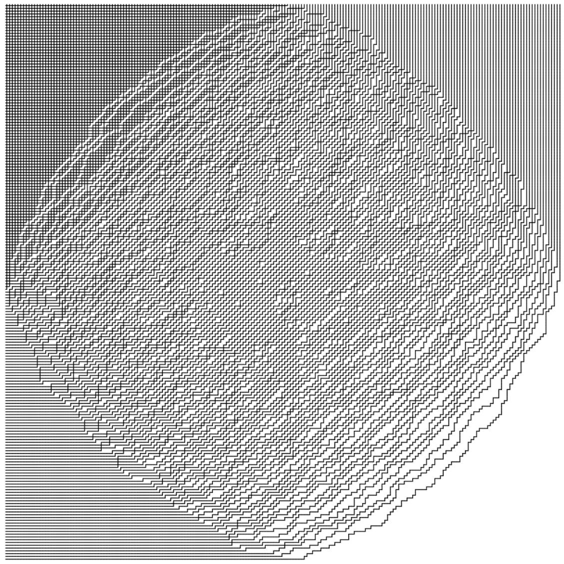
People also study models with more complicated weights.



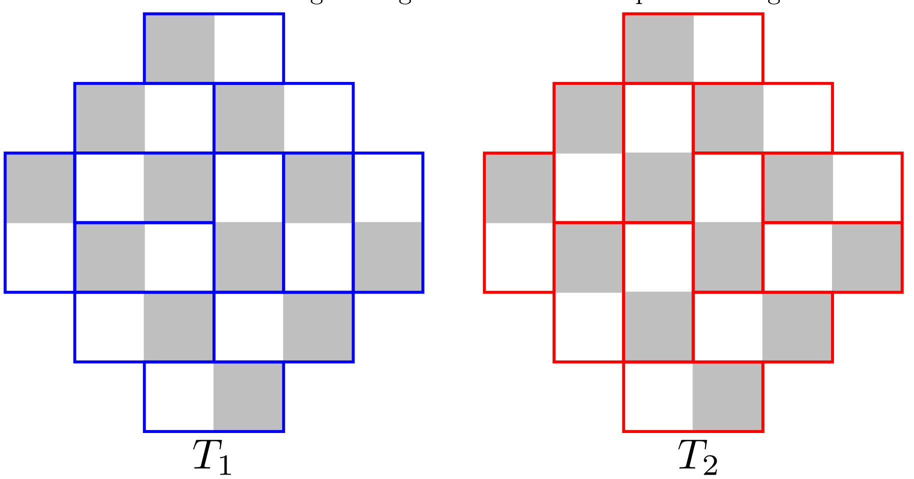
Aztec Diamond with peridoic weights (Berggren (2020))

The right image above is the arctic curve for that choice of weights on the AD. The interior "bubbles" are a different phase (not frozen or disordered) that does not appear in the Aztec diamond with uniform weights.

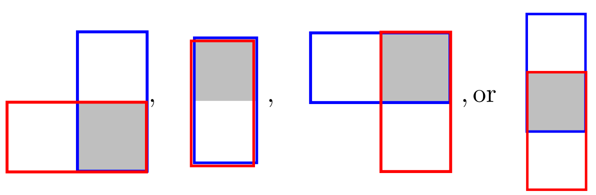
There are still many open questions pertaining to the six vertex model. Domino tilings are related to the six vertex model with a very specific choice of weights ($a = 1, b = 1, c = \sqrt{2}$). For general weights, understanding the asymptotics is very open.



Recently, tilings corresponding to color vertex models have been introduced. We will briefly discussed one I have worked on. Now rather than a single tiling we will consider a pair of tilings:



We will refer to the tilings as being different colors. Image the two tilings superimpose one on top of the other. Define 'interaction' by



Each 'interaction' give a power of t to the weight of the tiling.

- Might seem to come from no where, but there what you need to keep the integrability of the model.
- Really coming from a colored, Yang-Baxter integrable, vertex model.
- Related to LLT polynomials rather than Schur polynomials.

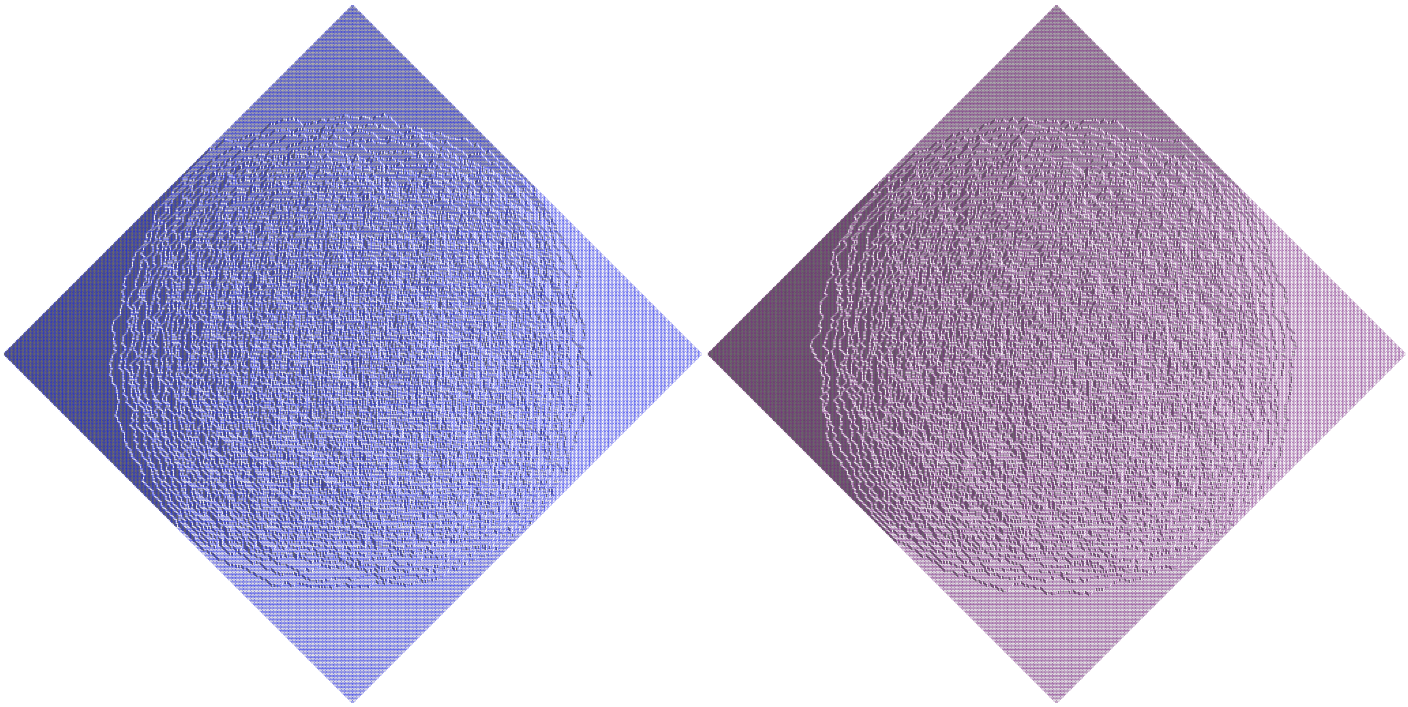
Define the partition function of the coupled tilings:

$$Z_N^{(2)}(t) = \sum_{\text{configs } \mathcal{C}} \text{weight}(\mathcal{C}) = \sum_{\text{configs } \mathcal{C}} t^{\#\text{interactions in } \mathcal{C}}$$

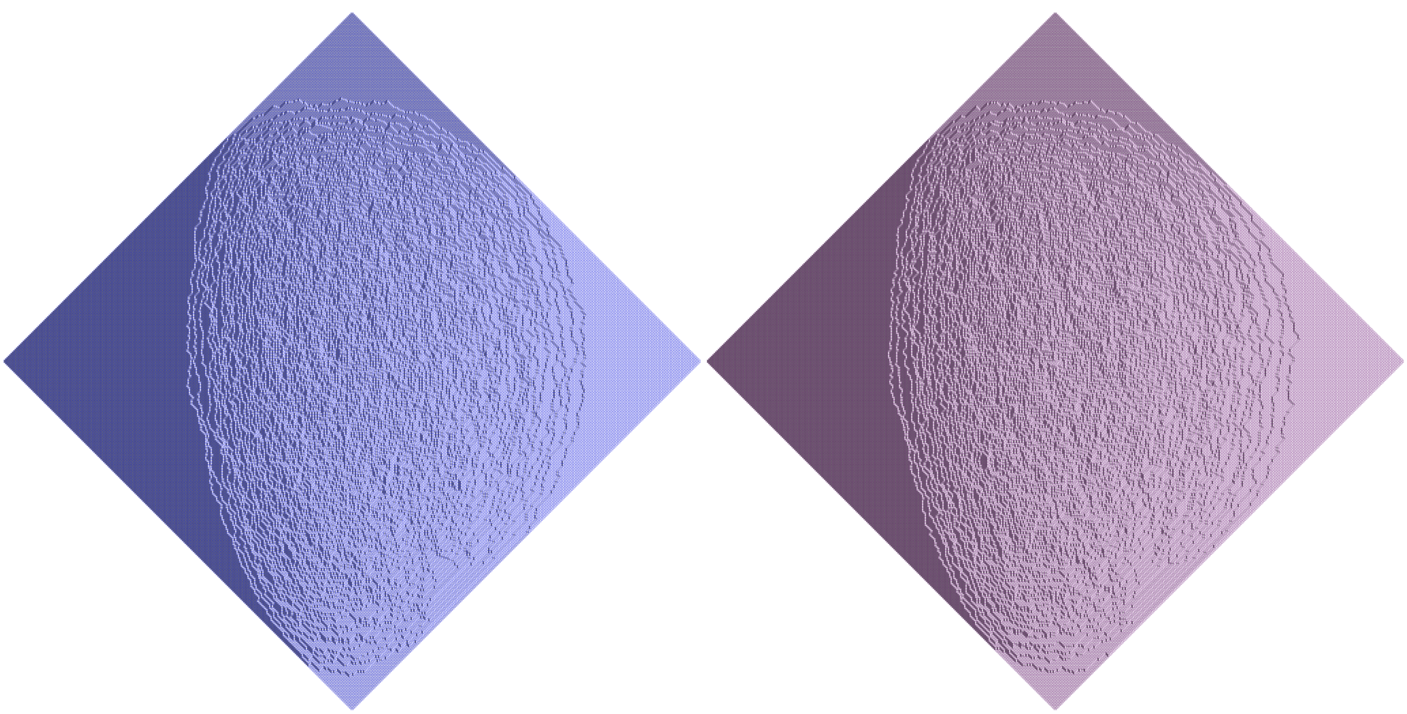
Theorem 0.25 (Corteel-Gitlin-K. (2022))**.** *The partition function for the coupled tilings of the Aztec diamond is given by*

$$Z_N^{(2)}(t) = (2(1+t))^{N(N+1)/2}$$

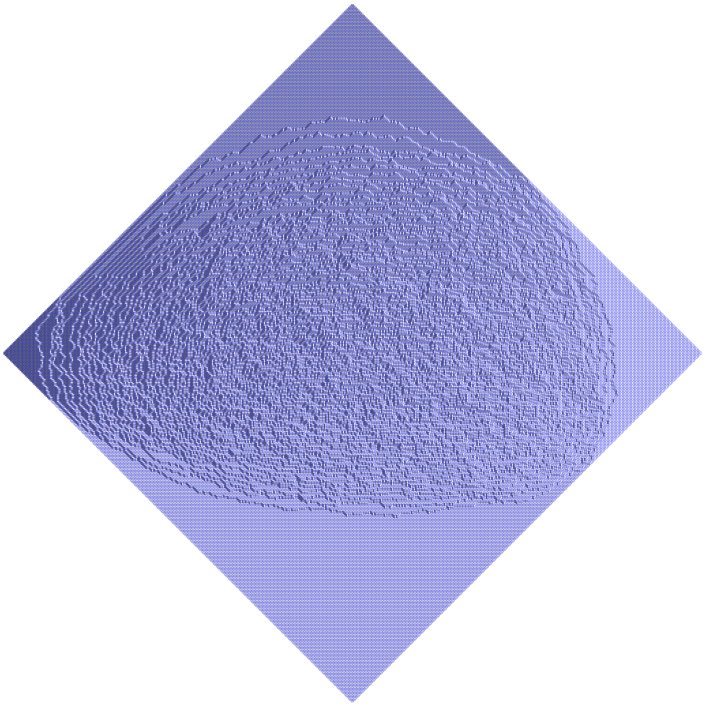
Now let's look at some large tilings.

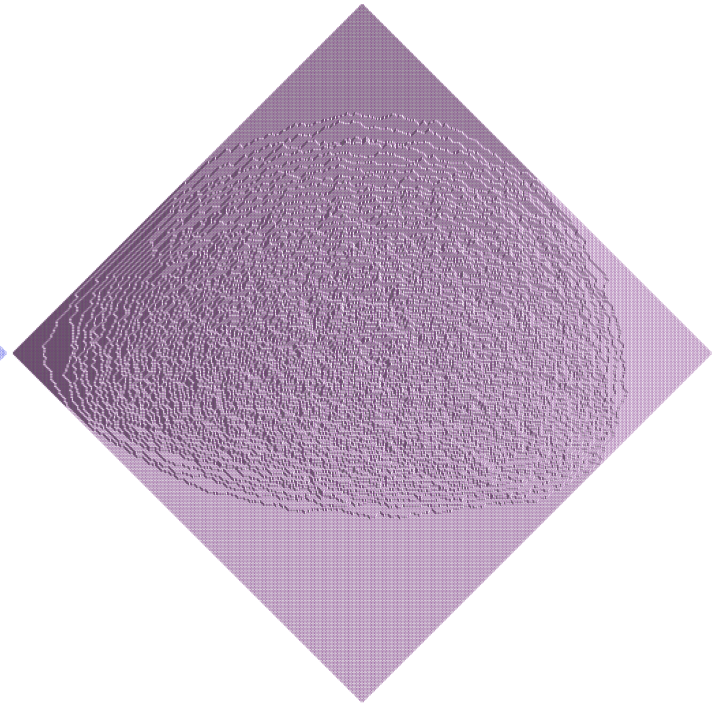


Here we have a 2-tiling of the rank-256 Aztec diamond at $t = 1$. In this case the tilings are independent and we see the expected circle. Now we turn on the interactions.

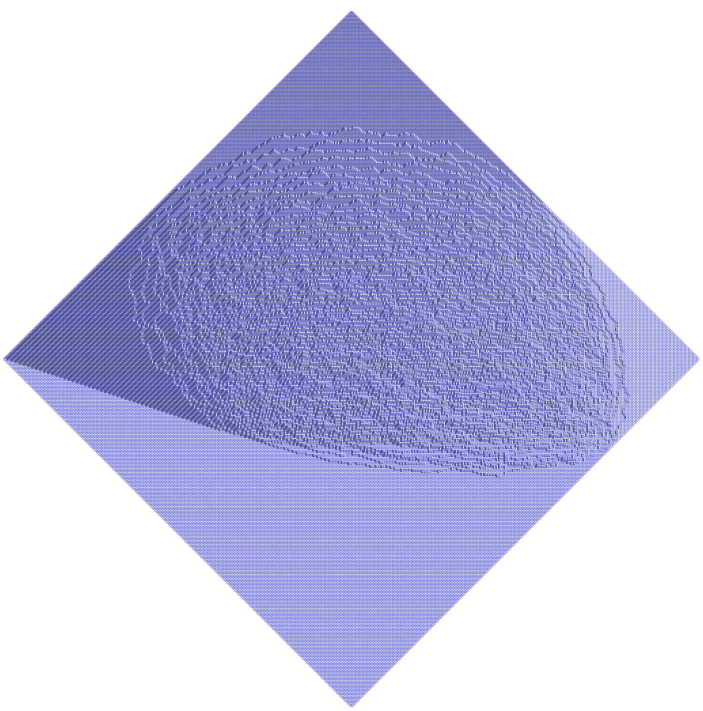


Above we have $t = 5$ while below we have $t = 0.2$.

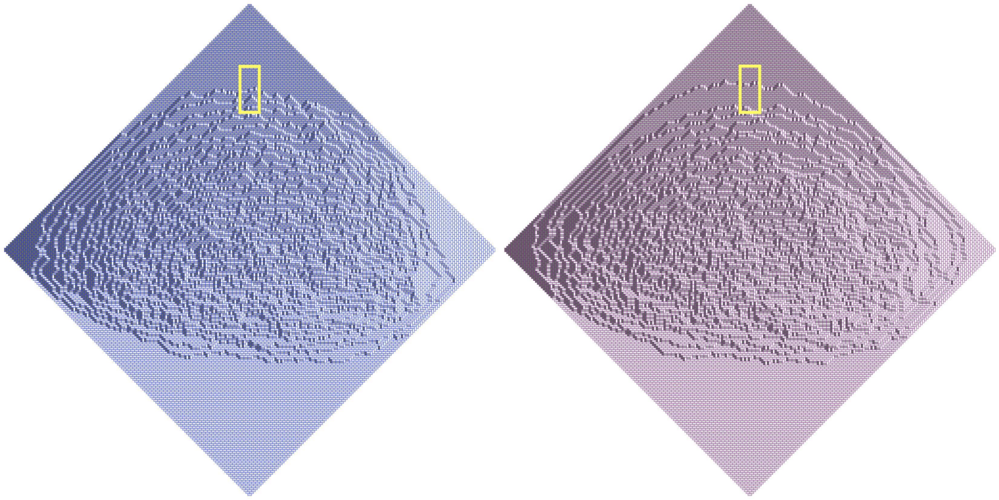


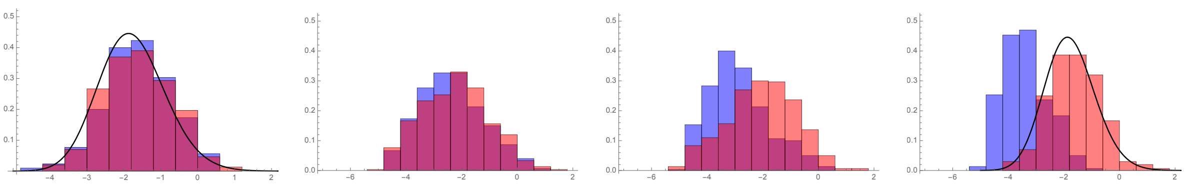


Finally, let's look at $t = 0$.



We can also look at the experimentally observed fluctuations.



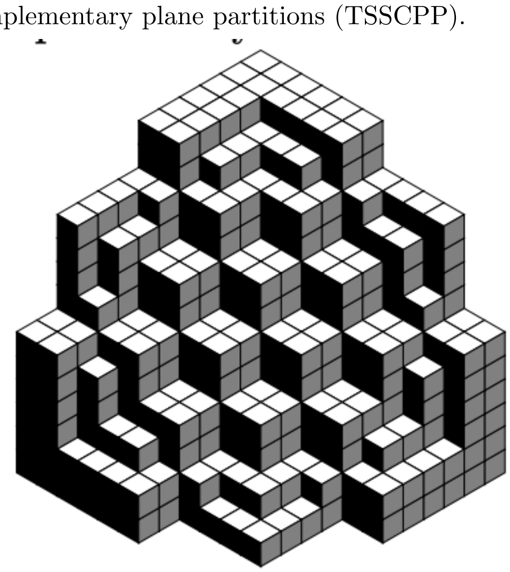


$t = 1$ $t = 0.5$ $t = 0.2$ $t = 0$

When $t = 1$ both colors have Tracy-Widom fluctuations, but when $t = 0$ only the red tiling has Tracy-Widom while the blue tiling fluctuates like the second largest eigenvalue of a GUE random matrix. For $t \in (0, 1)$ we somehow interpolate between these two extremes but how to describe these fluctuations is unknown.

To summarize, we see really interesting asymptotic phenomenon in the colored tilings! Understanding these phenomenon is still very much open.

Outside of asymptotics, there are still open enumeration questions as well. One well-known example is that of totally symmetric self complementary plane partitions (TSSCPP).



(Striker (2018))

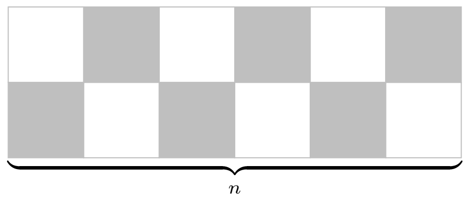
Andrews in 1994 proved that the number of such plane partitions in a $2n \times 2n \times 2n$ box is given by

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = ASM_n.$$

Its an open problem to give a bijective proof.

Lecture B9 Exercises.

Problem 1: Some domino tilings



$D_n = \#$ of tilings of a $2 \times n$ rectangle

(a) What is D_n? (Try some small cases, $n = 1, 2, 3, 4, ...$ and see if you notice a pattern.)

(b) Show that $D_n \sim \phi^n$ as $n \to \infty$ where $\phi = \frac{1+\sqrt{5}}{2}$. (One way to do this: Find a matrix M such that $\begin{pmatrix} D_n \\ D_{n-1} \end{pmatrix} = M \begin{pmatrix} D_2 \\ D_1 \end{pmatrix}$ and diagonalize.)