Numerics of Coupled Aztec Diamonds

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The Coupled Aztec Diamond

An Aztec Diamond (AD) of rank n is the union of squares in the plane such that $|x| + |y| < n$. This gives us a diamond shape. Note that there are 4 types of tiles: vertical and horizontal, each with 2 different ways they align on a checkerboard background. We can color them for easier visualization.

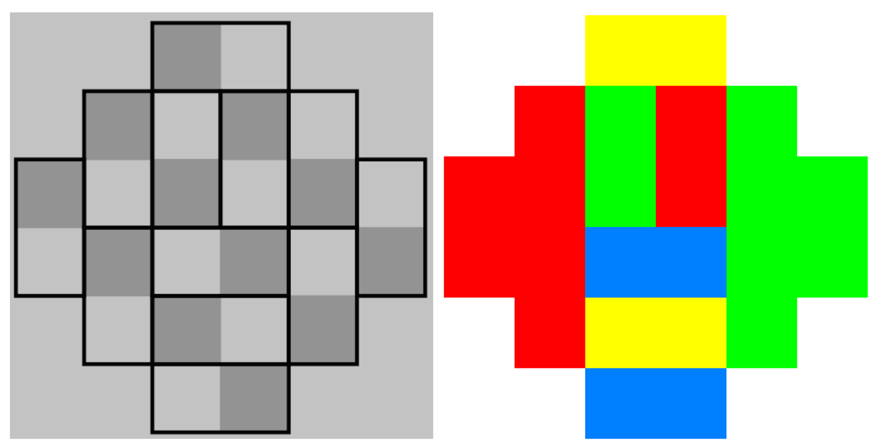

Fig. 1: AD of rank 3

For the uncoupled AD described above, there exists an algorithm to generate a tiling uniformly at random from all possible tilings [4]. This semester, we looked at coupled AD's. Instead of generating a single tiling, we generate two. We label one of them as the "smaller" tiling and we look at "interactions" between them.

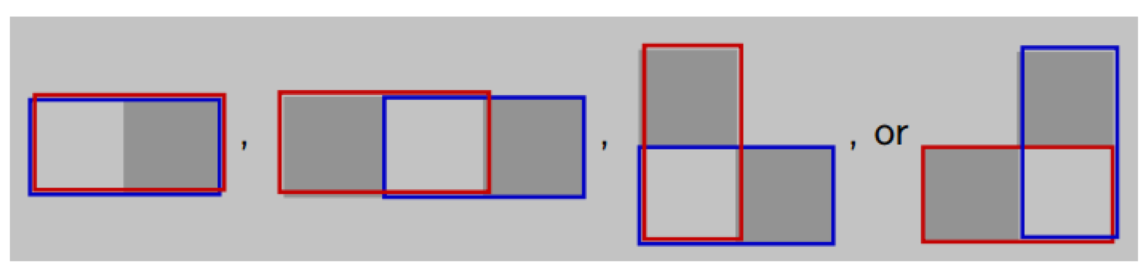

Fig. 2: Interactions. Smaller tiles are outlined in blue, bigger tiles in orange

Imagine placing the two tilings on top of each other and looking for these patterns. Each occurrence is an interaction. We now define some weighting parameter t and we consider generating two tilings, *not uniformly at random*, but with probability proportional to t^w where w is the number of interactions, or weight, that the two tilings have. [5] describes a slightly modified algorithm for generating coupled tilings. This gives us interesting shapes.

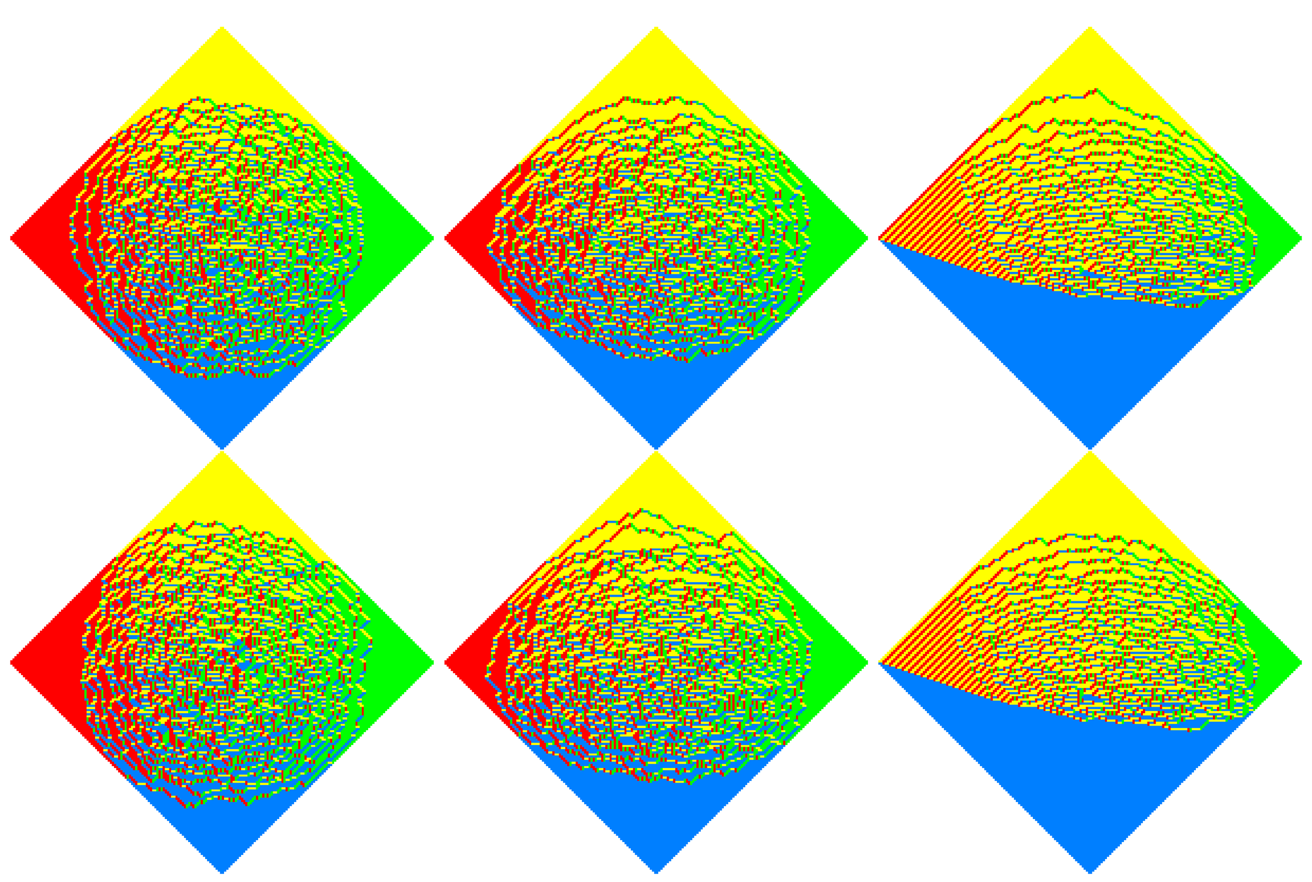

Fig. 3: Rank 100 Tilings at t=1,2,∞. Bigger tilings on top

Note that the $t = 1$ is exactly the *uncoupled* AD. Much is known about this, such as that the disordered region in the middle approaches a circle as the rank of the AD goes to ∞. The $t = 0, \infty$ cases are very restrictive. They are so restrictive, in fact, that a bijection exists between a single uncoupled AD and a pair of AD's at $t = 0$ or $t = \infty$ [2]. This allows us to to have known results about the $t = 0, \infty$ cases as well. The $t = 0$ case looks symmetric to $t = \infty$ if flipped across the line $y = x$, and likewise for $t = a$ and $t = 1/a$.

Edge Fluctuations

In Fig. 3 we see that around the edges of the unordered region there are distinct "paths". In fact, there is a bijection between paths and tilings. We expect these paths to move around a little for random tilings. In particular, if we define $X(0)$ to be the distance, in number of squares, from the center of the AD up to the top-most path, the distribution of $X(0)$ has been shown to converge to the Tracy-Widom (TW) F_2 distribution after some scaling and the rank goes to infinity. In fact, the ith top path under the same scaling appears to converge to the F_2^i distribution [3]. We can also look at the maximum "height" of the paths over their entire trajectories. The distribution of these maxima has been shown to converge to the TW F_1 distribution after some scaling and the rank goes to infinity. In fact, taking the max of the ith top path appears to converge to the F_1^i distribution with some analytically as-yet-undetermined offset, experimentally found to be around 1.3, 2.2, and 3.1 for the second, third, and fourth paths. These offsets are not included in the below plots.

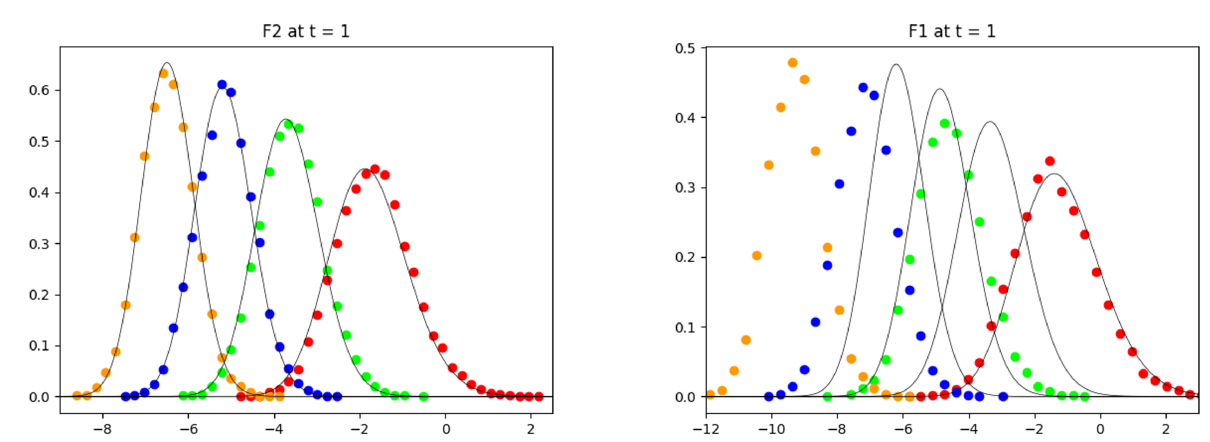

Fig. 4: 10,000 trials with uncoupled rank 500 AD. F_2^{1-4} and F_1^{1-4}

We can look at the interactions in coupled tilings in terms of their paths. The four interactions may seem arbitrary, but they actually serve to, in the cases of $t = 0, \infty$, force the paths of the smaller and larger tilings to not cross. This allows us to translate results about the uncoupled AD's paths into results about the limiting $t = 0, \infty$ cases. In particular, the small and larger tilings alternate paths, so we get the same distributions associated with different paths.

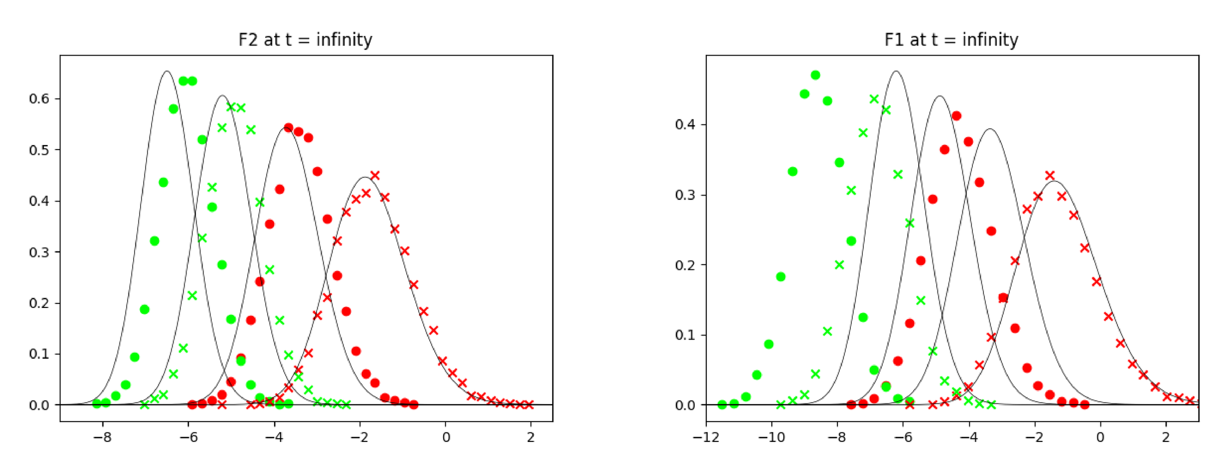

Fig. 5: 10,000 trials with rank 500 AD, $t = \infty$. F_2^{1-4} and F_1^{1-4}. Circles are smaller tiling, X's are bigger tiling

There is some error in the plots above; they are not quite aligned with the true distributions seen at $t = 1$. This is due to finite size AD's - it is computationally intensive to generate large rank AD's! Very interesting results occur when we pick t between 1 and ∞. Very little is known about these. Below is $t = 2$.

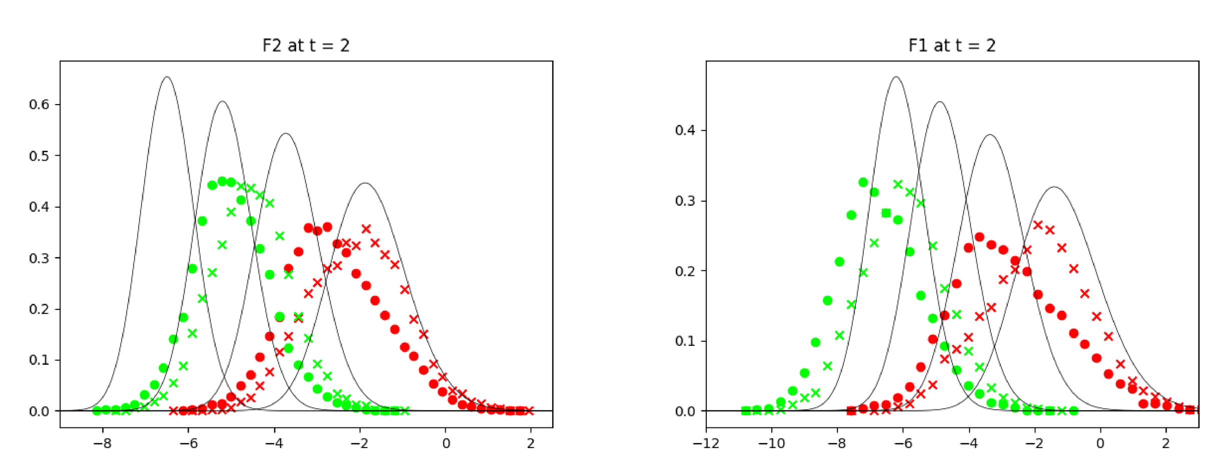

Fig. 6: 10,000 trials with rank 500 AD, $t = 2$. F_2^{1-4} and F_1^{1-4}. Circles are smaller tiling, X's are bigger tiling

Path Covariance

The top paths of the tilings is related to the Airy Process. A lot is known about this object. In fact, the Tracy-Widom distributions we see in the edge fluctuations are related to the Airy Process. Another result from the Airy Process that we computationally checked is the covariance between the middle of the top path and points near the middle. For points very near the middle, their covariance should decay linearly.

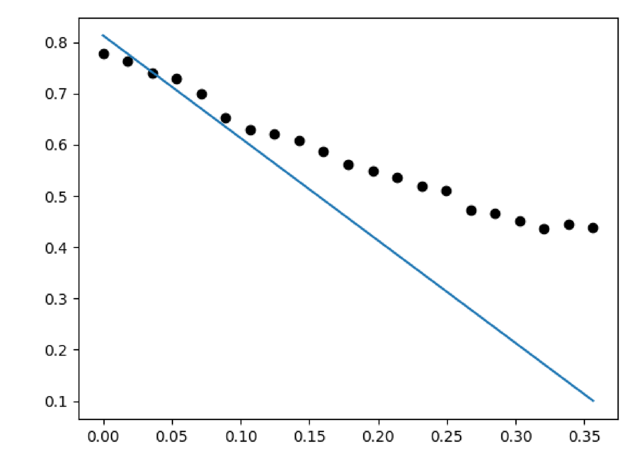

Fig. 7: Covariance. Predicted linear trend shown. 200 trials with rank 500 AD, $t = 1$.

Summary

We were able to experimentally verify many results associated with the uncoupled Aztec Diamond and, through the bijection, results about coupled Aztec Diamonds at $t = 0, \infty$. We generated experimental results for some cases where $t \neq 0, 1, \infty$.

A new results is one regarding the F_1^i series of distributions relating to the ith top path. They appear to follow the F_1^i distributions but with some offset.

One of the most surprising results was the $t = 2$ case, particularly the larger tiling's topmost path. At the $t = 1, \infty$ cases it lines up exactly with the F_2 and F_1 distribution, but at intermediary values it does not. Future work could explore these cases, either theoretically or experimentally.

Acknowledgements

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References

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