Numerical Simulations of Interacting Particle Systems

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Introduction

Goal / Motivation

We study particle processes such as TASEP and TAZRP, where particles evolve through exponential clocks and local jump rules.

These systems exhibit large-scale behavior captured by hydrodynamic limits and structured fluctuations.

We use simulation to track height functions, particle positions, fluctuations and compare them with theoretical predictions.

Our goal is to understand how these stochastic dynamics generate macroscopic structure.

TASEP

- **TASEP (Totally Asymmetric Simple Exclusion Process):** Particles evolve on a 1D lattice with at most one particle per site.
- Each particle has an independent exponential clock with rate 1. When the clock rings, the particle attempts to jump one site to the right.
- A jump occurs only if the destination site is empty, otherwise the jump is blocked.
- **Step Initial Condition:** At time $t = 0$, all sites $x \le 0$ are occupied and all sites $x > 0$ are empty.

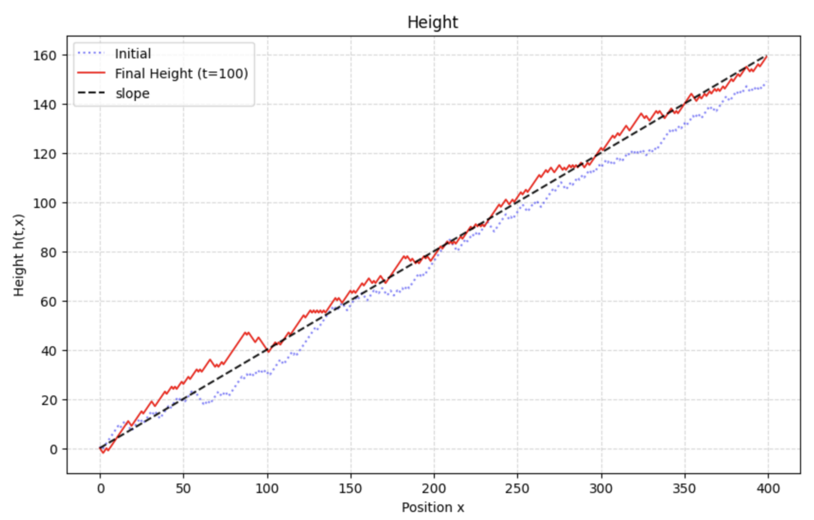
Hydrodynamic Limit / Height Function

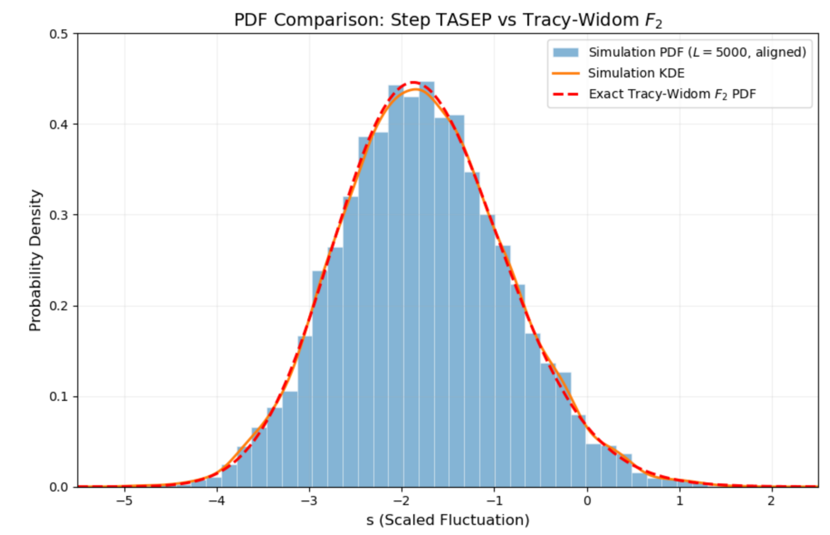
The height function $h(t, x)$ encodes the particle configuration by assigning down steps to particles and up steps to holes, with $h(0, x) = |x|$ under step initial conditions.

The hydrodynamic limit describes the large-scale behavior of the system. After rescaling,

$$\lim_{L\to\infty} \frac{1}{L} h(tL, xL) = \begin{cases} \dfrac{x^2+t^2}{2t}, & |x| \le t, \\ |x|, & |x| > t. \end{cases}$$

Thus, the random height function converges to a deterministic limit shape.





Fluctuations

We are also interested in the fluctuations around the hydrodynamic limit. A fluctuation refers to the random deviation of the observed height function from the deterministic limit. Under the same scaling as before, these fluctuations vanish, so we introduce a different scaling to obtain a non-trivial limit. We define the rescaled height function so that fluctuations remain visible in the limit:

$$h_L(t, x) = \frac{1}{L^{1/3}} h(Lt, L^{2/3}x) - L^{2/3}\frac{t}{2}.$$

An important observation is that these fluctuations do not follow a Gaussian distribution. Instead, they converge to the Tracy–Widom distribution, which describes the distribution of the largest eigenvalue of a GUE random matrix. At the origin $x = 0$, we obtain:

$$\lim_{L\to\infty} \mathbb{P}(h_L(1, 0) \ge -s) = F_{\mathrm{GUE}}(s),$$

where $F_{\mathrm{GUE}}(s)$ is the cumulative distribution function of the Tracy–Widom distribution.

References

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Part I: q-TAZRP

Model / Rules

- Totally Asymmetric Zero Range Process (TAZRP): Particles only jump to the right. When the clock at a site rings, if the site contains a nonzero number of particles, one particle jumps one space to the right.
- Unlike TASEP, infinitely many particles are allowed at each site. Each site has an independent exponential clock.
- Zero-range: the jump rate depends only on the number of particles at that site, not on neighboring sites.
- q-TAZRP: Each site has an exponential clock with rate

$$\frac{1-q^{\#\text{ of particles}}}{1-q}.$$

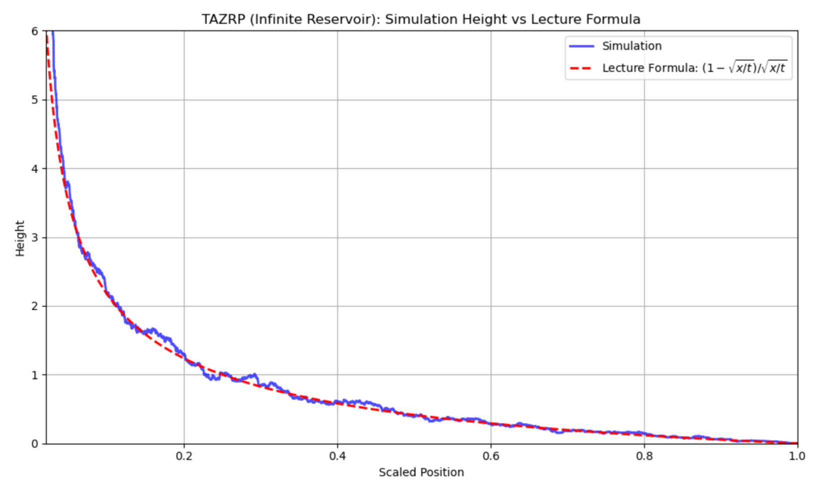
When it rings, if the site contains a nonzero number of particles, one of the particles jumps one space to the right. Note if $q = 0$ this reduces to TAZRP.

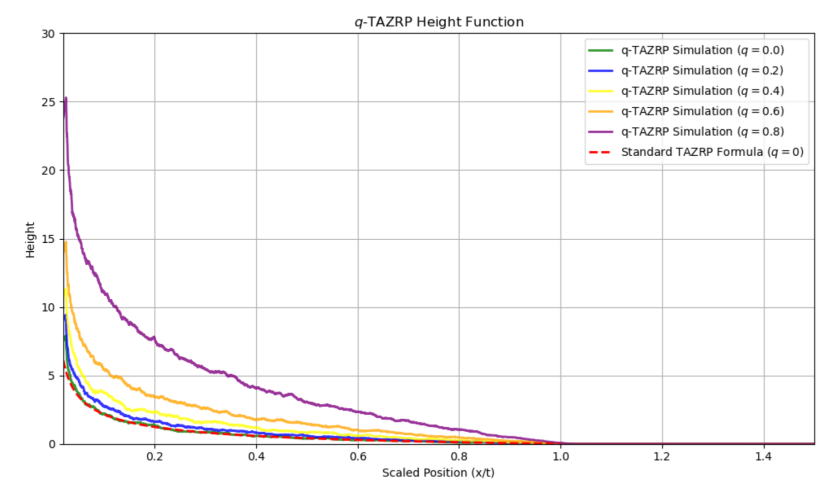
- Step initial condition: a single site with infinitely many particles immediately to the left of zero (reservoir), and all sites to the right are empty.

Height Functions and q-Dependence

- If there are k particles at a site we draw a line segment with slope $-k$ with $\lim_{x\to\infty} h(t, x) = 0$.
- Let $h(t, x)$ be the TAZRP height function. If we start from step initial condition then we have the hydrodynamic limit for TAZRP.

$$\lim_{L\to\infty} \frac{1}{L} h(tL, xL) = \begin{cases} \dfrac{1-\sqrt{x/t}}{\sqrt{x/t}}, & 0 < x \le t \\ 0, & \text{o.w.} \end{cases}$$





Fluctuations

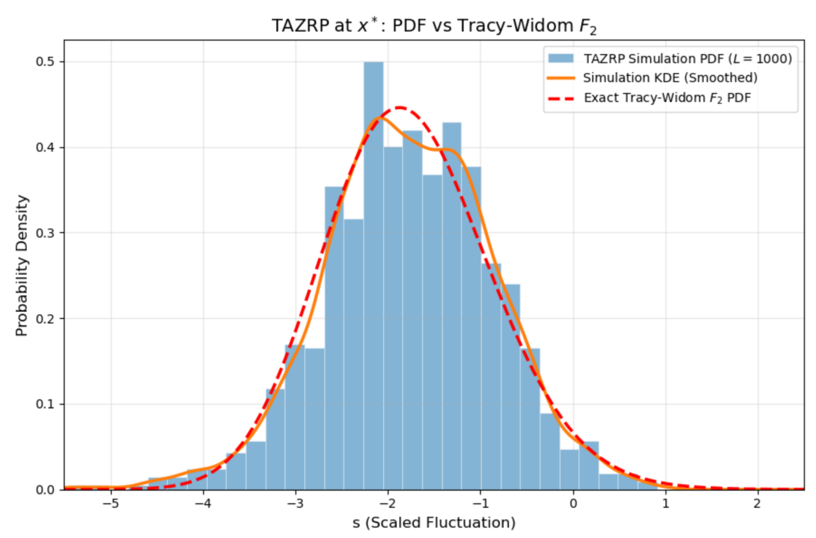
- We focus on the fluctuations of the intersection between the TAZRP height function and the main diagonal.
- To capture non-trivial fluctuations, we apply the scaling (the constant $\mathcal{C}$ here depends on the specific macroscopic evolution rate of TAZRP on the main diagonal):

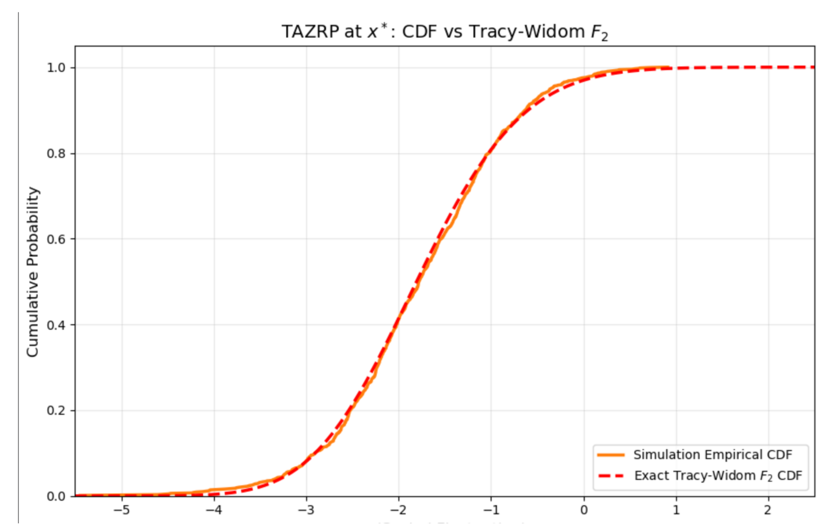
$$h_L(t, x) = \frac{1}{L^{1/3}} h(Lt, L^{2/3}x) - \mathcal{C} \cdot L^{2/3} t$$

- Non-Gaussian Behavior: As $L \to \infty$, the fluctuations converge to F_{GUE}:

$$\lim_{L\to\infty} \mathbb{P}(h_L(1, 0) \ge -s) = F_{\mathrm{GUE}}(s)$$

- Conclusion: Under step initial conditions, the rescaled fluctuations converge to the Tracy-Widom F_2 distribution.





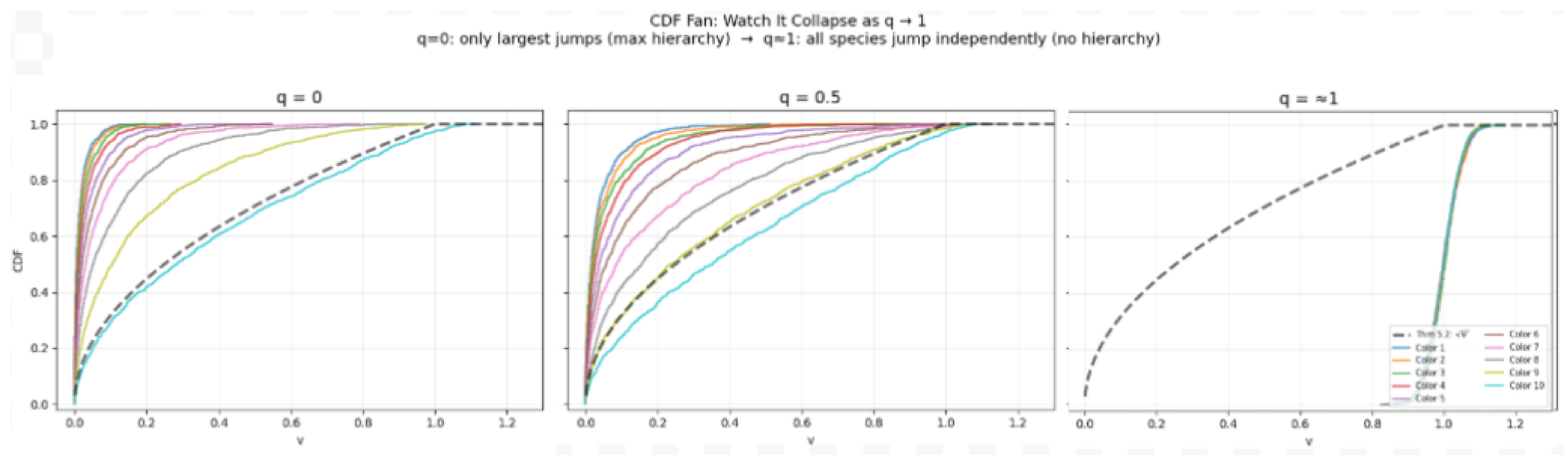
Part II: Multispecies

One Particle of Each Color

We simulate the **multispecies** q**-TAZRP** with $N = 10$ species labeled 1 to 10. At each site, particles of species n jump right at rate

$$q^{\#\text{species}>n} \cdot \frac{1-q^{\#\text{species } n}}{1-q}.$$

When species are ignored, the total jump rate at a site reduces to $\frac{1-q^{\#\text{particles}}}{1-q}$, recovering single-species q-TAZRP. Setting $q = 0$, only the largest species present jumps (rate 1). As $q \to 1$, each species n jumps independently at rate equal to its count. We place one particle of each color $1, \ldots, 10$ at the same site, with a reservoir of color 11 (highest priority) behind them. Since all 10 colors share sites as they evolve, they compete for jumps: higher-priority colors jump first and suppress lower ones through the $q^{\#\text{species}>n}$ prefactor. We simulate for $q \in \{0, 0.5, \approx 1\}$ and track each color's rescaled position $x_n(t)/t$.

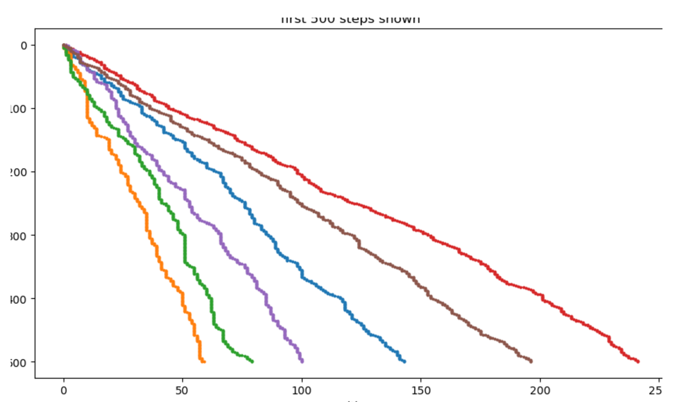


With 10 competing species, the priority hierarchy controlled by q determines each color's speed. At $q = 0$ only the highest-priority color moves freely; lower colors are suppressed. As q increases toward 1, the hierarchy flattens and all colors converge to the same speed, showing the CDF fan collapse.

Our Exploration: Rank-Dependent Model

Own exploration. We introduce a one-layer toy process, separate from the multispecies q-TAZRP, to isolate the effect of rank ordering. Here t is a rank-bias parameter. At each step, particles are sorted from right to left; after re-sorting, rank i jumps with probability $p_i(t) = t^i/(1 + t^i)$.

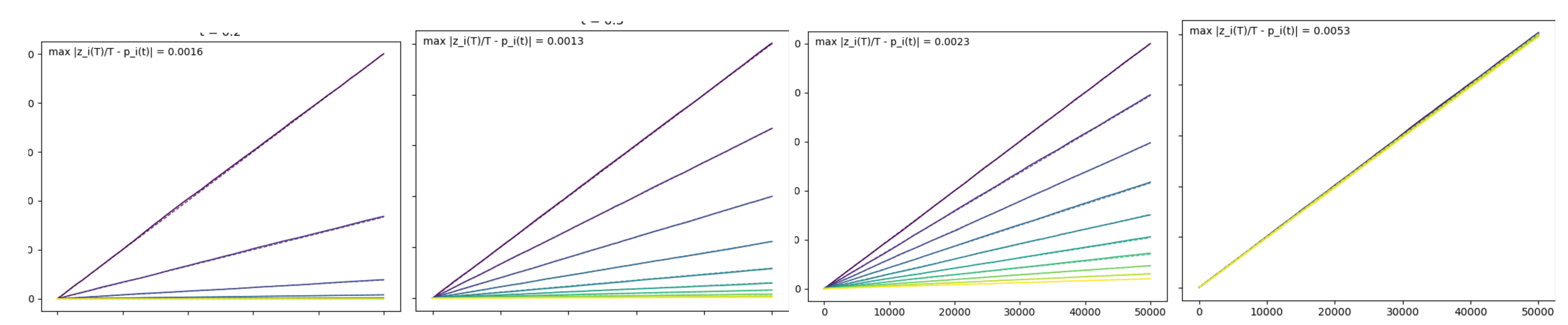
Built-in mechanism. For $t < 1$, $p_0(t) > p_1(t) > \cdots > p_{k-1}(t)$, so the rank rule predicts separated sample paths.



Simulation from our toy rank process.

Simulation Check: Rank-Sorted Slope Fan

Numerical check. Let $z_i(n)$ be the rank-i sorted trajectory in the toy process. We test whether its empirical slope matches the chosen jump probability, $z_i(n)/n \approx p_i(t)$, equivalently $z_i(n) \approx p_i(t)n$. Thus the fan below is a simulation check for our comparison model.



Toy-model check: solid curves are simulations; dashed curves are $p_i(t)n$. At $t = 1$, all ranks have slope $1/2$.

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