NOTES ON BIJECTIVIZATION

1. Basics of Bijectivization

To start with we need two finite sets $\mathcal{A}$ and $\mathcal{B}$. We'll suppose that each these sets come with a *weight* function $\mathbf{w}_A : \mathcal{A} \to \mathbb{R}_{>0}$ and $\mathbf{w}_B : \mathcal{B} \to \mathbb{R}_{>0}$, respectively, and that they satisfy the equality

$$\sum_{a\in\mathcal{A}} \mathbf{w}_A(a) = \sum_{b\in\mathcal{B}} \mathbf{w}_B(b). \tag{1.1}$$

Another way to describe this equality is to say the partition functions of each set are equal.

While Eqn. (1.1) tells us that the sum of the weights are equal, we would like more precise information, in particular, where does each term on the LHS contribute to the RHS and vice-versa. This is where bijectivization comes in. To be precise, a *bijectivization* is a collection of maps $\mathbf{p}^{fwd}$, $\mathbf{p}^{bwd} : \mathcal{A} \times \mathcal{B} \to \mathbb{R}_{\geq 0}$, which we write as $\mathbf{p}^{fwd}(a \to b)$ and $\mathbf{p}^{bwd}(b \to a)$, such that

- They satisfy the sum-to-one property

$$\begin{aligned} \sum_{b\in\mathcal{B}} \mathbf{p}^{fwd}(a \to b) &= 1 \quad \text{for all} \quad a \in \mathcal{A} \\ \sum_{a\in\mathcal{A}} \mathbf{p}^{bwd}(b \to a) &= 1 \quad \text{for all} \quad b \in \mathcal{B}. \end{aligned} \tag{1.2}$$

- They satisfy the reversibility condition

$$\mathbf{w}_A(a)\mathbf{p}^{fwd}(a \to b) = \mathbf{w}_B(b)\mathbf{p}^{bwd}(b \to a). \tag{1.3}$$

We call $\mathbf{p}^{fwd}$ the *forward maps* and $\mathbf{p}^{bwd}$ the *backward maps*.

Remark 1.1. The first bullet point tells us that we should think of these maps as transition probabilities for certain Markov chains. That is, $\mathbf{p}^{fwd}(a \to b)$ tells us the probability that starting from a we go to b, while $\mathbf{p}^{bwd}(b \to a)$ tells us the probability that starting from b we go to a. The second bullet point tells us that these Markov chains are in some sense dual to one another.

Note that a bijectivization gives us a refinement of Eqn. (1.1). We now have

$$\sum_{a\in\mathcal{A}} \mathbf{w}_A(a) = \sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}} \mathbf{w}_A(a)\mathbf{p}^{fwd}(a \to b) = \sum_{b\in\mathcal{B}}\sum_{a\in\mathcal{A}} \mathbf{w}_B(b)\mathbf{p}^{bwd}(b \to a) = \sum_{b\in\mathcal{B}} \mathbf{w}_B(b).$$

where the first and last equalities follow from the sum-to-one property (1.2) and the middle equality follows from the reversibility condition (1.3). One can think of $\mathbf{p}^{fwd}(a \to b)$ as telling us what fraction of $\mathbf{w}_A(a)$ on the LHS contributes to $\mathbf{w}_B(b)$ on the RHS.

It is important to note that in general bijectivization is not unique, as we will see in the following example.

Example 1.2. Let $\mathcal{A} = \{a_1, a_2\}$ and $\mathcal{B} = \{b_1, b_2\}$. Suppose our weight functions are such that

$$\mathbf{w}_A(a_1) = \mathbf{w}_A(a_2) = \mathbf{w}_B(b_1) = \mathbf{w}_B(b_2) = 1.$$

In this case the reversibility condition (1.3) tells us that

$$\mathbf{p}^{fwd}(a_i \to b_j) = \mathbf{p}^{bwd}(b_j \to a_i)$$

for each $i, j \in \{1, 2\}$.

Suppose $r_1, r_2 \in (0, 1)$. Let's choose

$$\mathbf{p}^{fwd}(a_1 \to b_1) = r_1 \quad \text{and} \quad \mathbf{p}^{fwd}(a_2 \to b_2) = r_2.$$

By the sum-to-one property (1.2), we then must have

$$\mathbf{p}^{fwd}(a_1 \to b_2) = 1 - r_1 \quad \text{and} \quad \mathbf{p}^{fwd}(a_2 \to b_1) = 1 - r_2.$$

For the backward maps, the sum-to-one property gives us the constraint on r_1 and r_2 that

$$\mathbf{p}^{bwd}(b_1 \to a_1) + \mathbf{p}^{bwd}(b_1 \to a_2) = r_1 + 1 - r_2 = 1$$
$$\mathbf{p}^{bwd}(b_2 \to a_1) + \mathbf{p}^{bwd}(b_2 \to a_2) = 1 - r_1 + r_2 = 1$$

This is satisfied if $r_1 = r_2$.

We see that for any $r \in (0, 1)$ defining our forward and backward maps as

$$\mathbf{p}^{fwd}(a_i \to b_j) = \begin{cases} r, & \text{if } i = j \\ 1 - r, & \text{o.w.} \end{cases}$$

$$\mathbf{p}^{bwd}(b_j \to a_i) = \begin{cases} r, & \text{if } i = j \\ 1 - r, & \text{o.w.} \end{cases}$$

is a valid choice of bijectivization.

However, in some special cases, there is a unique bijectivization. Clearly, if $|\mathcal{A}| = |\mathcal{B}| = 1$ then for Eqn. (1.1) to be true we must have $\mathbf{w}_A(a) = \mathbf{w}_B(b)$ and the only choice of bijectivization is $\mathbf{p}^{fwd}(a \to b) = 1 = \mathbf{p}^{bwd}(b \to a)$.

For a more complicated case we have the following.

Proposition 1.3. *If $|\mathcal{A}| = 1$ then there is a unique bijectivization given by*

$$\mathbf{p}^{fwd}(a \to b) = \frac{\mathbf{w}_B(b)}{\sum_{b' \in \mathcal{B}} \mathbf{w}_B(b')}$$

and

$$\mathbf{p}^{bwd}(b \to a) = 1$$

for $a \in \mathcal{A}$ and all $b \in \mathcal{B}$.

If $|\mathcal{B}| = 1$ then there is a unique bijectivization given by

$$\mathbf{p}^{fwd}(a \to b) = 1$$

and

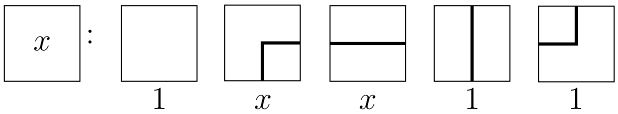
$$\mathbf{p}^{bwd}(b \to a) = \frac{\mathbf{w}_A(a)}{\sum_{a' \in \mathcal{A}} \mathbf{w}_A(a')}$$

for $b \in \mathcal{B}$ *and all* $a \in \mathcal{A}$.

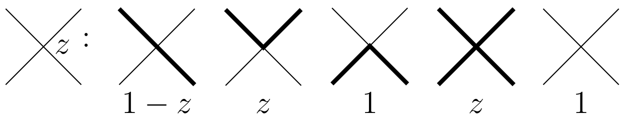
Exercise 1. *Prove Prop. 1.3. (Difficulty rating: 2)*

2. Bijectivizing the Yang-Baxter Equation

2.1. **Review of the 5-vertex model.** Recall the five-vertex model whose local configurations and weights are given by

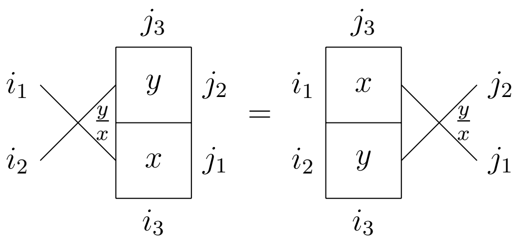


with any other local path configuration having weight zero. We will also need the cross vertices



where below we also write the corresponding weight. Together the vertices satisfy what is known as the *Yang-Baxter equation* (YBE):

Proposition 2.1. *For any choice of boundary condition* $i_1, i_{2,3}, j_1, j_2, j_3 \in \{0, 1\}$ *(with* 0 *indicating there is no path, and* 1 *indicating there is a path), we have the equality of partition functions*

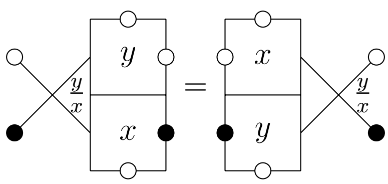


(2.1)

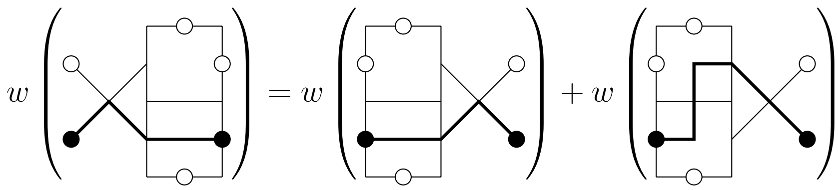
where x *and* y *are the weight parameters at each vertex.*

We will think of paths as always moving to the right or upward, so we call (i_1, i_2, i_3) the *entering* paths and (j_1, j_2, j_3) the *exiting* paths. The YBE can be stated in other words as fixing where paths enter and exit the domains to be the same on both sides of the equation, then the sum of the weights of the allowed path configurations on both sides is equal. Consider the following example.

Example 2.2. Suppose we set $i_2 = j_1 = 1$ (the black dots) and the rest to zero (the white dots). Then the YBE says we have the following equality of partition functions:



Explicitly writing the sum over path configurations, this becomes



where we see there are two ways to fill in the paths on the left but only one on the right. Computing the weights we have

$$\text{LHS:} \qquad x$$

$$\text{RHS:} \qquad y + x \cdot \left(1 - \frac{y}{x}\right) = x$$

which are, in fact, equal.

2.2. **Applying bijectivization.** Note that the YBE (2.1) gives an equality for any choice of entering and exiting paths. Each of these equations can be seen as an equality of sums of weights over finite sets. That is, for any choice of entering and exiting paths we can try to bijectivize the equality coming from the YBE.

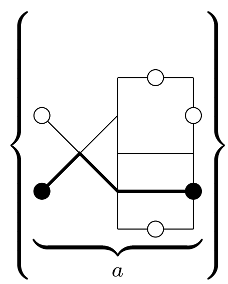
To make this more precise, let $\mathcal{A}(i_1, i_2, i_3; j_1, j_2, j_3)$ be the set of allowed path configurations for the domain on the LHS of the YBE (2.1) with boundary conditions given by (i_1, i_2, i_3) for the entering paths and (j_1, j_2, j_3) for the exiting paths. Similarly, let $\mathcal{B}(i_1, i_2, i_3; j_1, j_2, j_3)$ be the set of allowed path configurations for the domain on the RHS of the YBE (2.1). These sets come with weight functions $\mathbf{w}_A$ and $\mathbf{w}_B$, respectively, given by the vertex model weights. The YBE ensures that Eqn. (1.1)

$$\sum_{a \in \mathcal{A}(i_1,i_2,i_3;j_1,j_2,j_3)} \mathbf{w}_A(a) = \sum_{b \in \mathcal{B}(i_1,i_2,i_3;j_1,j_2,j_3)} \mathbf{w}_B(b)$$

is satisfied for each choice of $i_1, i_2, i_3, j_1, j_2, j_3$. So we may try the bijectivize the equation by finding appropriate transition probabilities $\mathbf{p}^{fwd}_{x,y}$ and $\mathbf{p}^{bwd}_{x,y}$, where the subscripts are there to explicitly denote that the forward and backward maps will depend on the vertex weights.

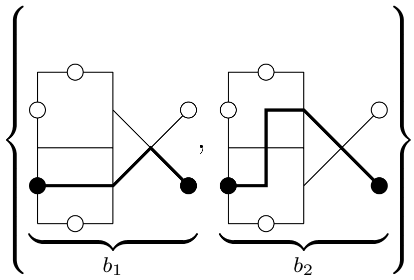
Example 2.3. Let us continue Example 2.2. In this case, we have

$\mathcal{A}(0, 1, 0; 1, 0, 0) =$



and

$$\mathcal{B}(0,1,0;1,0,0) =$$



Note that $|\mathcal{A}(0,1,0;1,0,0)| = 1$, so we may apply Prop. 1.3 and get

$$\mathbf{p}^{fwd}_{x,y}(a \to b_1) = \frac{\mathbf{w}(b_1)}{\mathbf{w}(b_1)+\mathbf{w}(b_2)} = \frac{y}{x}$$

$$\mathbf{p}^{fwd}_{x,y}(a \to b_2) = \frac{\mathbf{w}(b_2)}{\mathbf{w}(b_1)+\mathbf{w}(b_2)} = 1 - \frac{y}{x}$$

and

$$\mathbf{p}^{bwd}_{x,y}(b_1 \to a) = \mathbf{p}^{bwd}_{x,y}(b_2 \to a) = 1.$$

This is a valid bijectivization as long as $0 < \frac{y}{x} < 1$.

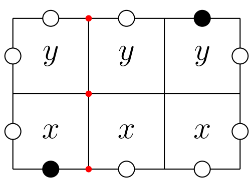
Exercise 2. *Naively, there are $2^6 = 64$ possible boundary conditions one can choose in the YBE. However, for most of these boundary conditions there are no possible valid path configurations. In fact, there are only 16 choices that do not result in the trivial equality $0 = 0$. List all 16 of these boundary conditions. (Difficulty rating: 1)*

Exercise 3. *Show that for any of the boundary conditions, $i_1, i_2, i_3, j_1, j_2, j_3$, you found in Exercise 2 either $|\mathcal{A}(i_1,i_2,i_3;j_1,j_2,j_3)| = 1$ or $|\mathcal{B}(i_1,i_2,i_3;j_1,j_2,j_3)| = 1$ (or both). Thus, for our five-vertex model Yang-Baxter equation, there is always a unique choice of bijectivization. (Difficulty rating: 1)*

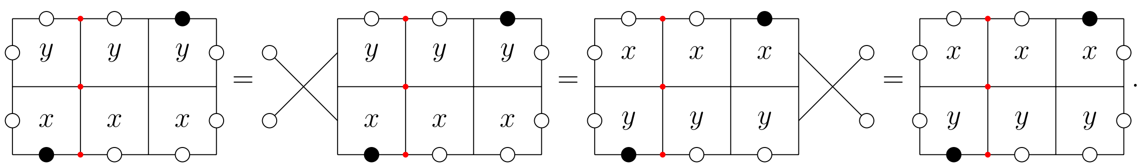
Exercise 4. *For any of the boundary conditions, $i_1, i_2, i_3, j_1, j_2, j_3$, you found in Exercise 2 determine $\mathbf{p}^{fwd}_{x,y}$. (Difficulty rating: 1)*

2.3. **Row swapping.** To push this further, consider the following example.

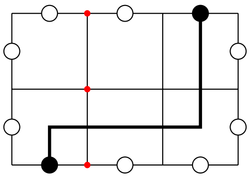
Example 2.4. Consider of the following pair of rows



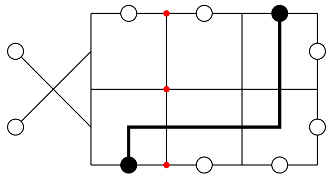
where the bottom boundary is given by the partition $\mu = (0)$ and the top boundary is given by the partition $\lambda = (2)$. We may introduce an empty cross on the LHS, as it has weight 1, then repeatedly apply the YBE to swap the two rows, and finally remove an empty cross from the RHS. Pictorially we have the equality of partition functions



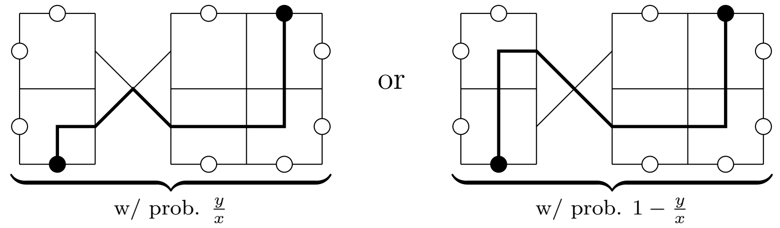
Suppose now we consider a specific configuration



with weight x^3. If we let ν be the partition describing the position of the path along the middle slice, in this case we have $\nu = (2)$. Just as before we may introduce an empty cross on the LHS

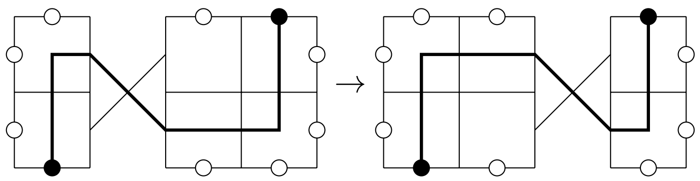


Now we use our bijectivized YBE (from Exercise 4) to get



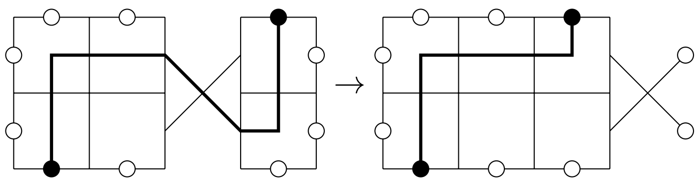
We can think of this as the path either staying in the bottom row (with probability $\frac{y}{x}$) or choosing to jump to the top row (with probability $1 - \frac{y}{x}$).

If the path chose to jump upward, then another step of the bijectivized YBE gives



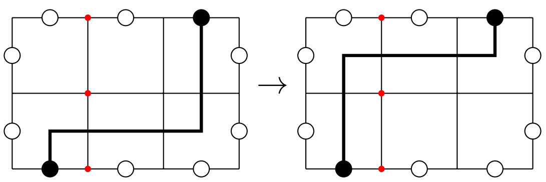
w/ prob. 1.

Another step gives



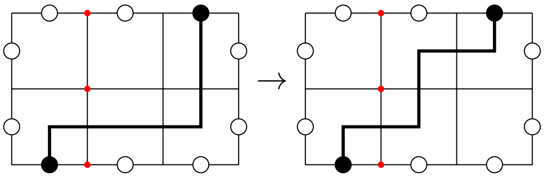
w/ prob. 1

and finally we can remove the empty cross from the RHS. Altogether we see that

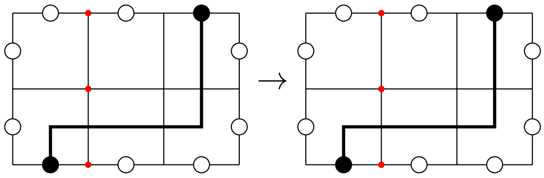


with probability $1 - \frac{y}{x}$.

If the path had instead chosen to stay in the bottom row, repeatedly applying the bijectivized YBE gives



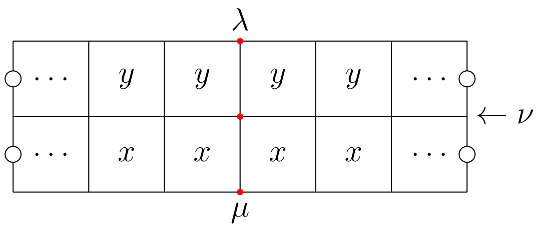
w/ prob. $\frac{y}{x}\left(1-\frac{y}{x}\right)$



w/ prob. $\left(\frac{y}{x}\right)^2$

Rather than the YBE just telling us that the partition function of the two rows remains unchanged after swapping, by bijectivizing we get a Markov chain telling us how the path configurations update after swapping.

In general, we have a pair of rows whose bottom boundary condition is given by partition μ, whose top boundary condition is given by a partition λ, and whose interior path configuration is determined by a partition ν.



First, let's set some notation. Let $Z_{\lambda/\mu}(x,y)$ be the partition function of two rows with bottom boundary μ, top boundary λ, parameter x in the bottom row, and parameter y in the top row. This is the partition function of the two rows above. After swapping the rows using the YBE, the partition function is $Z_{\lambda/\mu}(y,x)$ and we have

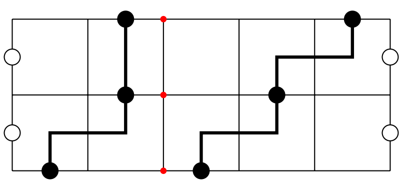
$$Z_{\lambda/\mu}(x,y) = Z_{\lambda/\mu}(y,x).$$

Note that we may write the partition function as

$$Z_{\lambda/\mu}(x,y) = \sum_{\nu} \mathbf{w}_{x,y}(\nu)$$

where the sum is over all partitions ν such that $\mu \preceq \nu \preceq \lambda$ and $\mathbf{w}_{x,y}(\nu)$ is the weight of the path configuration whose paths cross the interior slice as locations determined by the partition ν.

Example 2.5. Fix $\mu = (1,0)$ and $\lambda = (3,1)$. If $\nu = (2,1)$ then the corresponding path configuration is



w/ weight x^2y.

The partition function for the two rows is given by

$$\begin{aligned}&Z_{\lambda/\mu}(x,y)\\&=\mathbf{w}_{x,y}((1,0))+\mathbf{w}_{x,y}((2,0))+\mathbf{w}_{x,y}((3,0))+\mathbf{w}_{x,y}((1,1))+\mathbf{w}_{x,y}((2,1))+\mathbf{w}_{x,y}((3,1))\\&=y^3+xy^2+x^2y+xy^2+x^2y+x^3.\end{aligned}$$

The *Gibbs measure* is a probability measure on the possible path configurations such that the probability of getting a particular path configuration is proportional to its weight. In particular, we have

$$\mathbb{P}_{x,y}(\nu|\lambda,\mu)=\frac{\mathbf{w}_{x,y}(\nu)}{Z_{\lambda/\mu}(x,y)}. \tag{2.2}$$

Swapping the rows using the bijectivized YBE determines transition probabilities for the paths along the middle slice to go from ν to another partition ν'. Let $\mathbf{P}_{x,y}(\nu\to\nu'|\lambda,\mu)$ be the transition probability from ν to ν' given fixed top and bottom boundary conditions given by λ and μ, respectively.

Exercise 5. *Fix $\mu=(1,0)$ and $\lambda=(3,1)$. Determine*

$$\mathbf{P}_{x,y}((2,1)\to(2,0)|\lambda,\mu).$$

(Difficulty rating: 2).

Exercise 6. *Fix $\mu=(0)$, $\lambda=(k)$. For each $\ell,\ell'\in\{0,\dots,k\}$ determine*

$$\mathbf{P}_{x,y}((\ell)\to(\ell')|\lambda,\mu).$$

(Difficulty rating: 2).

Exercise 7. *Show that $\mathbf{P}_{x,y}(\nu\to\nu'|\lambda,\mu)$ is only nonzero iff $\mu\preceq\nu'\preceq\nu\preceq\lambda$. (Difficulty rating: 2)*

The transition probabilities we get from bijectivization behave nicely with the Gibbs measures in the following sense.

Proposition 2.6. *Bijectivization preserves Gibbs measures. That is, if the initial configuration is sampled according to the Gibbs measure $\mathbb{P}_{x,y}(\cdot|\lambda,\mu)$ then the probability that, after swapping, the path configuration is determined by ν' is given by $\mathbb{P}_{y,x}(\nu'|\lambda,\mu)$.*

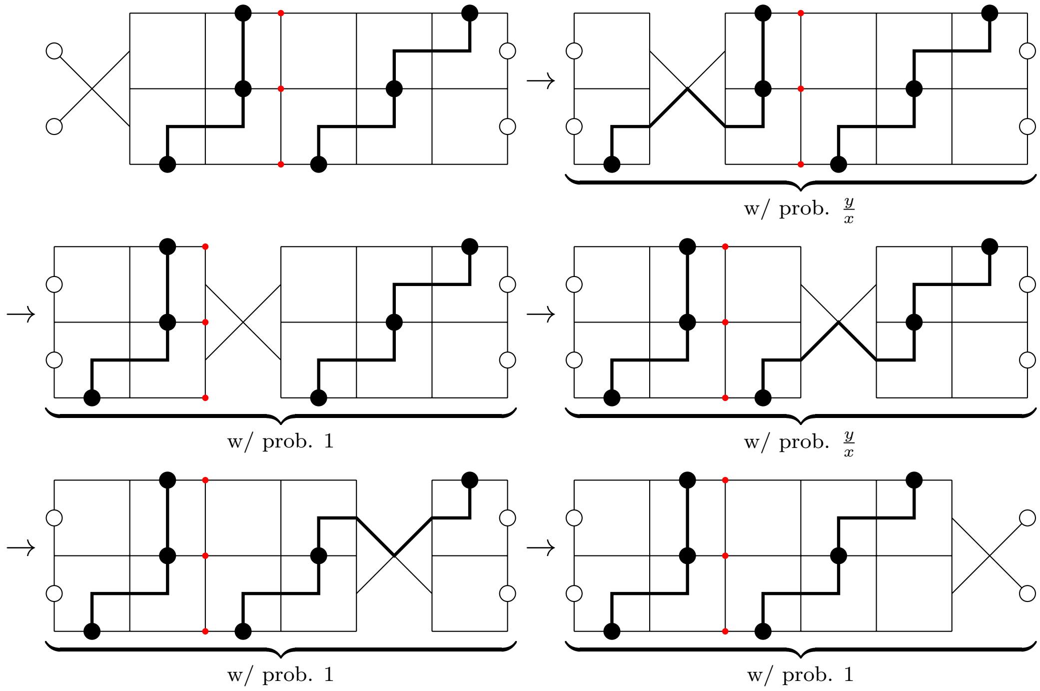
Let's see this in action in an example.

Example 2.7. Fix $\mu=(1,0)$ and $\lambda=(3,1)$. Suppose that the the initial configuration is sampled from the Gibbs measure $\mathbb{P}_{x,y}(\cdot|\lambda,\mu)$. Let's compute the probability that after swapping we end up with the configuration $\nu'=(2,1)$.

First note that there are only two possible starting partitions that can transition to $\nu'=(2,1)$: either $\nu=(2,1)$ or $\nu=(3,1)$. We have

$$\mathbb{P}(\nu')=\mathbf{P}_{x,y}((2,1)\to(2,1)|\lambda,\mu)\mathbb{P}_{x,y}((2,1)|\lambda,\mu)+\mathbf{P}_{x,y}((3,1)\to(2,1)|\lambda,\mu)\mathbb{P}_{x,y}((3,1)|\lambda,\mu).$$

Now let's compute each of the transition probabilities. If $\nu = (2,1)$ we get



We see that $\mathbf{P}_{x,y}((2,1) \to (2,1)|\lambda,\mu) = \left(\frac{y}{x}\right)^2$. A similar calculation shows that $\mathbf{P}_{x,y}((3,1) \to (2,1)|\lambda,\mu) = \left(\frac{y}{x}\right)^2\left(1-\frac{y}{x}\right)$. Plugging these in we get

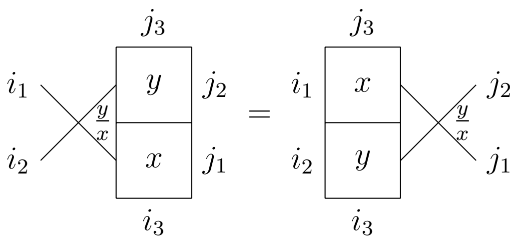
$$\begin{aligned}\mathbb{P}(\nu') &= \left(\frac{y}{x}\right)^2 \mathbb{P}_{x,y}((2,1)|\lambda,\mu) + \left(\frac{y}{x}\right)^2\left(1-\frac{y}{x}\right)\mathbb{P}_{x,y}((3,1)|\lambda,\mu) \\ &= \left(\frac{y}{x}\right)^2 \frac{x^2y}{Z_{\lambda/\mu}(x,y)} + \left(\frac{y}{x}\right)^2\left(1-\frac{y}{x}\right)\frac{x^3}{Z_{\lambda/\mu}(x,y)} \\ &= \frac{y^3+xy^2-y^3}{Z_{\lambda/\mu}(x,y)} \\ &= \frac{xy^2}{Z_{\lambda/\mu}(y,x)}\end{aligned}$$

where in the last equality we use that $Z_{\lambda/\mu}(x,y) = Z_{\lambda/\mu}(y,x)$. Altogether, we have

$$\mathbb{P}(\nu') = \frac{xy^2}{Z_{\lambda/\mu}(y,x)} = \mathbb{P}_{y,x}((2,1)|\lambda,\mu)$$

as we expect from Prop. 2.6.

Sketch of proof of Prop. 2.6. The key calculation is as follows. Consider the YBE



As defined previously, let $\mathcal{A} := \mathcal{A}(i_1, i_2, i_3; j_1, j_2, j_3)$ be the set of allowed path configurations for the domain on the LHS and let $\mathcal{B} := \mathcal{B}(i_1, i_2, i_3; j_1, j_2, j_3)$ be the set of allowed path configurations for the domain on the RHS. Suppose the configuration on the LHS is sampled with probability proportional to its weight. The probability we get a particular configuration $b \in \mathcal{B}$ on the RHS is given by

$$\begin{aligned}\mathbb{P}(b) &= \sum_{a\in\mathcal{A}} \mathbb{P}(a)\mathbf{p}^{fwd}_{x,y}(a \to b)\\ &= \sum_{a\in\mathcal{A}} \frac{\mathbf{w}(a)\mathbf{p}^{fwd}_{x,y}(a \to b)}{Z_A(x,y)}\end{aligned}$$

where $Z_A(x,y) = \sum_{a\in\mathcal{A}} \mathbf{w}(a)$. Now using the reversibility condition (1.3), we may write this as

$$\mathbb{P}(b) = \sum_{a\in\mathcal{A}} \frac{\mathbf{w}(b)\mathbf{p}^{bwd}_{x,y}(b \to a)}{Z_A(x,y)}$$

The sum-to-one condition (1.2) gives

$$\begin{aligned}\mathbb{P}(b) &= \frac{\mathbf{w}(b)}{Z_A(x,y)}\\ &= \frac{\mathbf{w}(b)}{Z_B(y,x)}\end{aligned}$$

where in the last equality we use the fact that $Z_a = Z_B$ (which follows from the YBE). We see that if the configuration on the LHS is sampled with probability proportional to its weight then the probability we get a particular configuration on the RHS is also proportional to its weight.

Now in the general case, we repeat this type of calculation while pushing the cross through each column. At every step the probability of the resulting configuration is given by the corresponding Gibbs measure. Pushing the cross all the way to the right gives the proposition.

□