Arctic Curves for Bounded Lecture Hall Tableaux

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Outline

Lecture Hall Tableaux

Fix positive integer n. Given $\lambda = (\lambda_1, ..., \lambda_n)$ (some λ_i possibly zero). Consider tableaux T of shape λ satisfying

$$\frac{T_{i\,j}}{n-i+j} \geq \frac{T_{i\,j+1}}{n-i+(j+1)}$$
$$\frac{T_{i\,j}}{n-i+j} > \frac{T_{i+1\,j}}{n-(i+1)+j}$$

(See Corteel-Kim 2018.)

Lecture Hall Tableaux

- When λ has only one column, these become lecture hall partitions.
- When $n \to \infty$, the conditions become

$$T_{i\,j} \geq T_{i\,j+1}$$
$$T_{i\,j} > T_{i+1\,j}.$$

These are the same conditions as those for reverse semistandard young Tableaux.

Bounded LHT

Fix positive integers n, t. Given $\lambda = (\lambda_1, ..., \lambda_n)$ (some λ_i possibly zero). Consider tableaux T of shape λ satisfying

$$\frac{T_{i\,j}}{n-i+j} \geq \frac{T_{i\,j+1}}{n-i+(j+1)}$$
$$\frac{T_{i\,j}}{n-i+j} > \frac{T_{i+1\,j}}{n-(i+1)+j}$$
$$\frac{T_{i\,j}}{n-i+j} < t.$$

Call Z_λ^t the number of such tableaux.

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$$\frac{T_{i\,j}}{n-i+j} < t.$$

Call Z^t_λ the number of such tableaux.
Remark: $s_\lambda(t+y_1, \ldots, t+y_n) = \sum_{\mu\subset\lambda} Z^t_{\lambda/\mu} s_\mu(y_1, \ldots, y_n)$.
(See Corteel-Kim 2019.)

Bounded LHT

$$\lambda = (5,4,3,2,1),\ n = t = 5$$

24	23	25	24	25
16	18	17	18	
11	11	2		
6	2			
0				

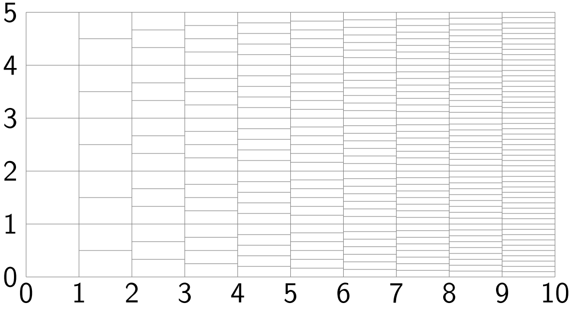
$$T_{ij}$$

$\frac{24}{5}$	$\frac{23}{6}$	$\frac{25}{7}$	$\frac{24}{8}$	$\frac{25}{9}$
$\frac{16}{4}$	$\frac{18}{5}$	$\frac{17}{6}$	$\frac{18}{7}$	
$\frac{11}{3}$	$\frac{11}{4}$	$\frac{2}{5}$		
$\frac{6}{2}$	$\frac{2}{3}$			
$\frac{0}{1}$				

$$\frac{T_{ij}}{n-i+j}$$

LHT as Nonintersecting Paths

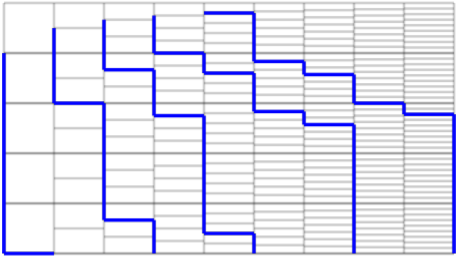
Consider the graph below ($t = 5$)



LHT as Nonintersecting Paths

(Here $n = 5, t = 5$)
Starting points: $v_i = (n - i, t - \frac{1}{n-i+1})$.
Ending points: $u_j = (n + \lambda_j - j, 0)$.
Row i of $T \leftrightarrow$ path from v_i to u_i.

24	23	25	24	25
16	18	17	18	
11	11	2		
6	2			
0				



Proposition (Corteel, Kim, Savage 18)

Fix n, t, λ. The number of LHT with $\frac{T_{ij}}{n-i+j} < t$ is given by

$$Z_\lambda^t = t^{|\lambda|} s_\lambda(\underbrace{1, \ldots, 1}_{n \text{ times}}) = t^{|\lambda|} \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Proposition (Corteel, Kim, Savage 18)

Fix n, t, λ. The number of LHT with $\frac{T_{ij}}{n-i+j} < t$ is given by

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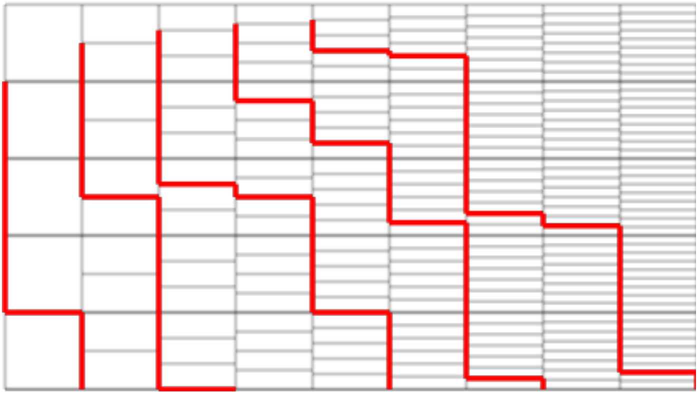
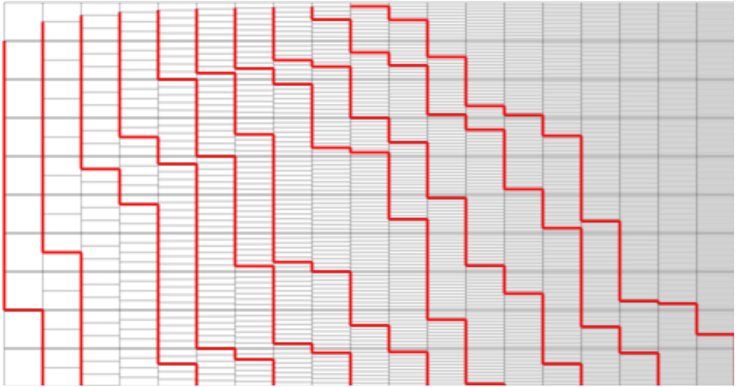
Proof.

Easy determinant evaluation. $\square$

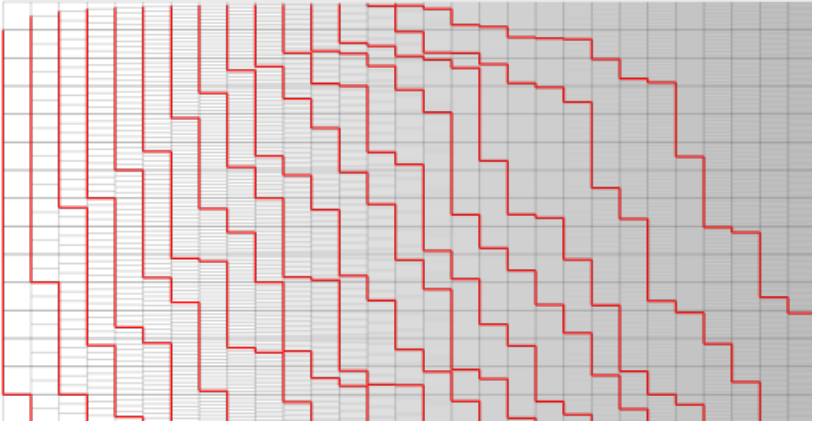
$$\lambda = (n, n-1, \ldots, 1)$$



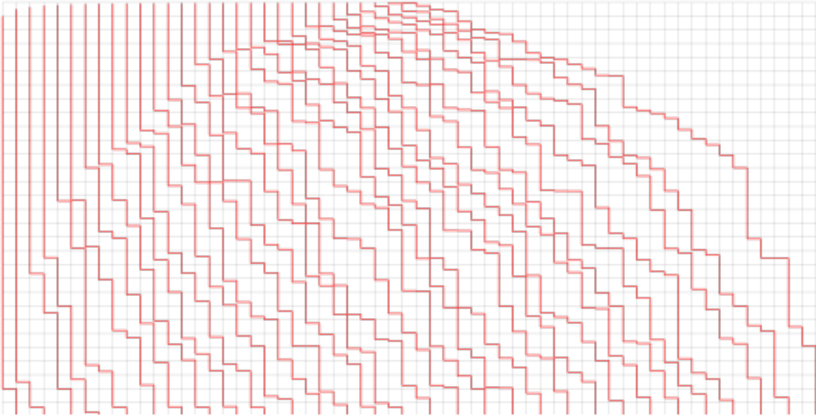
$n = t = 5$



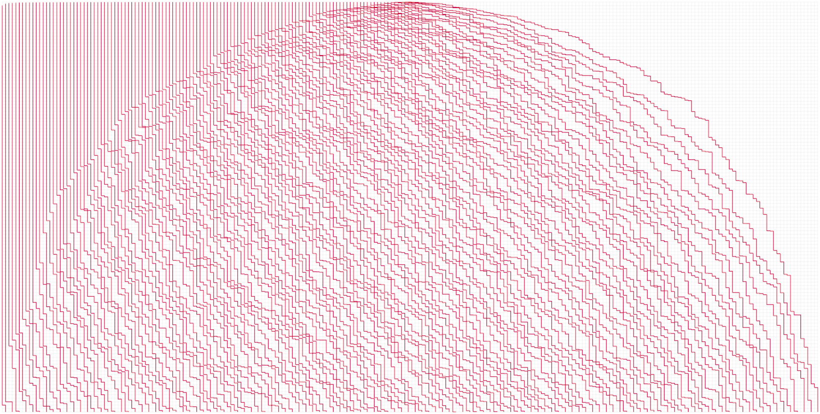
$$n = t = 10$$



$n = t = 15$



$n = t = 30$



$n = t = 120$

Some History

Tangent Method: A way to derive an arctic curve for systems that can be modeled as a collection of nonintersecting paths.

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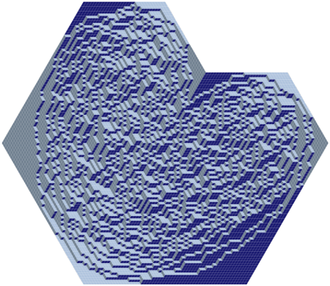
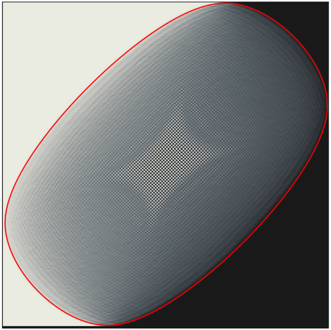
- 2005: Kenyon, Okounkov, "Limit shapes and the complex burgers equation."



Some History

Tangent Method: A way to derive an arctic curve for systems that can be modeled as a collection of nonintersecting paths.

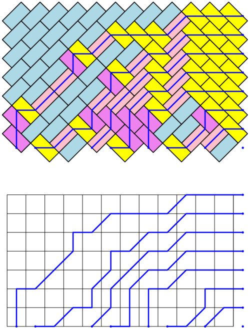
- 2016: Colomo, Sportiello, "Arctic curves of the six-vertex model on generic domains: the Tangent Method."



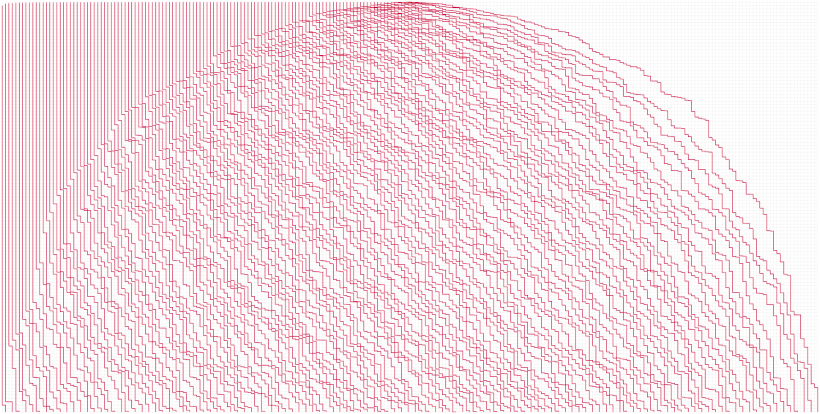
Some History

Tangent Method: A way to derive an arctic curve for systems that can be modeled as a collection of nonintersecting paths.

- 2016-now: Aggarwal, Debin, di Francesco, Granet, Guitter, Lapa, Ruelle, and others.



Tangent Method

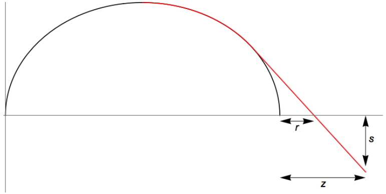


$n = t = 120$

Tangent Method

Idea:

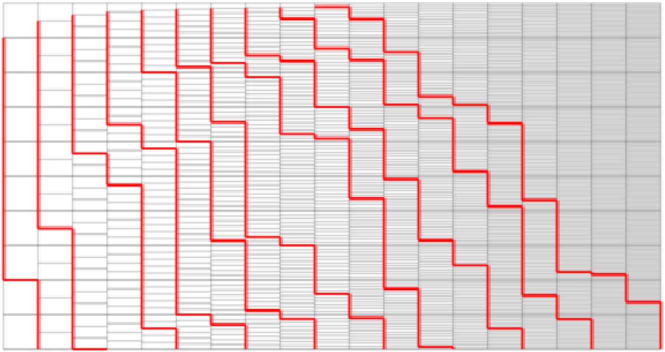
- In the thermodynamic limit, the outermost path follows the arctic curve.
- Extend outermost path by z, s as shown.



- Assumption: The path will follow the arctic curve until it can move in a straight line to its endpoint. The line is tangent to the arctic curve.
- Compute most probable r. The points $(n + \lambda_1 - 1 + r, 0)$ and $(n + \lambda_1 - 1 + z, -s)$ define the tangent line.
- Varying z gives a family of lines tangent to the arctic curve.

An Easy Example

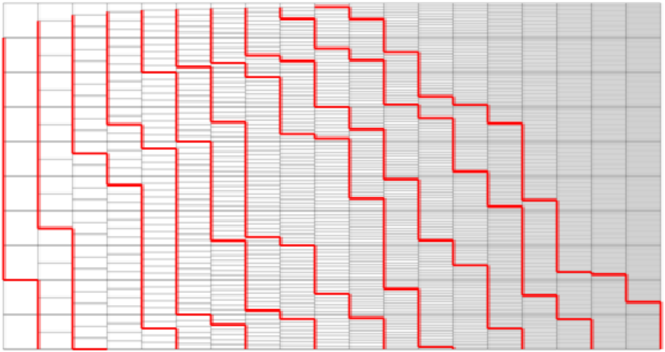
$$\lambda = (n, n-1, \ldots, 1)$$



$$n = 10,\ t = 10$$

$\lambda = (n, n-1, ..., 1)$

$$Z_\lambda^t = t^{|\lambda|} s_\lambda(1, \dots, 1) = (2t)^{\binom{n+1}{2}}$$

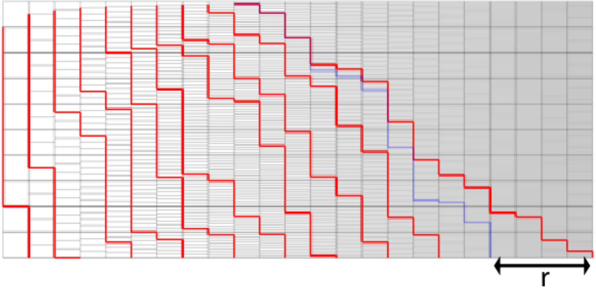


$n = 10, \; t = 10$

$\lambda = (n, n-1, \ldots, 1)$

$$\frac{Z^t_{\lambda,r}}{Z^t_\lambda} = t^r \prod_{j=2}^{n} \frac{\lambda_1 + r - \lambda_j + j - 1}{\lambda_1 - \lambda_j + j - 1} = t^r \binom{n+k-1}{k}$$

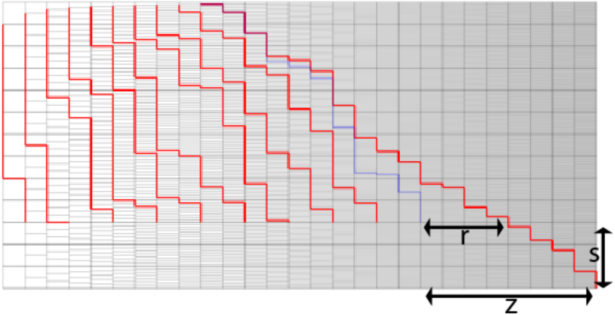
(for $r = 2k$)



$n = 10,\ t = 10,\ r = 4$

$\lambda = (n, n-1, \ldots, 1)$

$$\frac{Z^t_{\lambda,z,s}}{Z^t_\lambda} = \sum_{r=0}^{z} \frac{Z^t_{\lambda,r}}{Z^t_\lambda} s^{z-r} \binom{2n+z-1}{z-r}$$



$n = 10,\ t = 10,\ z = 8,\ s = 3$

$\lambda = (n, n-1, ..., 1)$

We take the limit $r = n\rho$, $z = n\zeta$, $t = n\tau$, $s = n\sigma$, and $n \to \infty$.

$$\frac{Z^t_{\lambda,z,s}}{Z^t_\lambda} \approx \frac{1}{2\pi} e^{\zeta\, n\, ln(n)} \int_0^\zeta \sqrt{\frac{1+\zeta}{\rho(\zeta-\rho)}} e^{n\, S(\rho)} d\rho$$

$$S(\rho) = \rho\, ln(\tau) + (\zeta-\rho) ln(\sigma) - \frac{1}{2}(2+\rho) ln(2+\rho) - \frac{1}{2}\rho\, ln(\rho) - (\zeta-\rho) ln(\zeta-\rho)$$

$$S(\rho) = \rho\, ln(\tau) + (\zeta - \rho) ln(\sigma) - \frac{1}{2}(2+\rho) ln(2+\rho) - \frac{1}{2}\rho\, ln(\rho) - (\zeta - \rho) ln(\zeta - \rho)$$

Has a unique maximum such that $S(\rho_{max}) = 0$.

$$\frac{Z^t_{\lambda,z,s}}{Z^t_\lambda} \propto e^{n\, S(\rho_{max})}$$

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Has a unique maximum such that $S(\rho_{max}) = 0$.

$$\frac{Z^t_{\lambda,z,s}}{Z^t_\lambda} \propto e^{n\,S(\rho_{max})}$$

Let's find ρ_{max}.

$\lambda = (n, n-1, ..., 1)$

$$S'(\rho) = 0 \implies \zeta - \rho = \frac{\sigma}{\tau}(2+\rho)\sqrt{\frac{\rho}{2+\rho}}$$

$\lambda = (n, n-1, \ldots, 1)$

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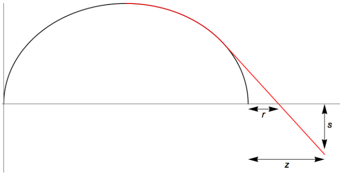
Pair of points: $(2+\rho, 0)$ and $(2+\zeta, -\sigma)$.



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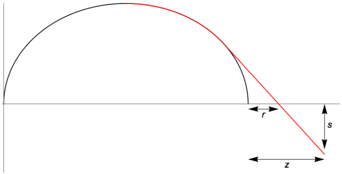

Family of tangent lines

$$Y = -\frac{\tau}{x}\sqrt{\frac{x}{x-2}}(X-x)$$

parametrized by $x = 2+\rho$, $x \in [2, \infty)$.

$\lambda = (n, n-1, ..., 1)$

Family of tangent lines:

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parametrized by $x = 2 + \rho$, $x \in [2, \infty)$.
Parametrization:

$$X(x) = \frac{x}{x-1}$$
$$Y(x) = \tau\frac{x}{x-1}\sqrt{\frac{x-2}{x}}$$

$\lambda = (n, n-1, \dots, 1)$

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Parametrization:

$$X(x) = \frac{x}{x-1}$$

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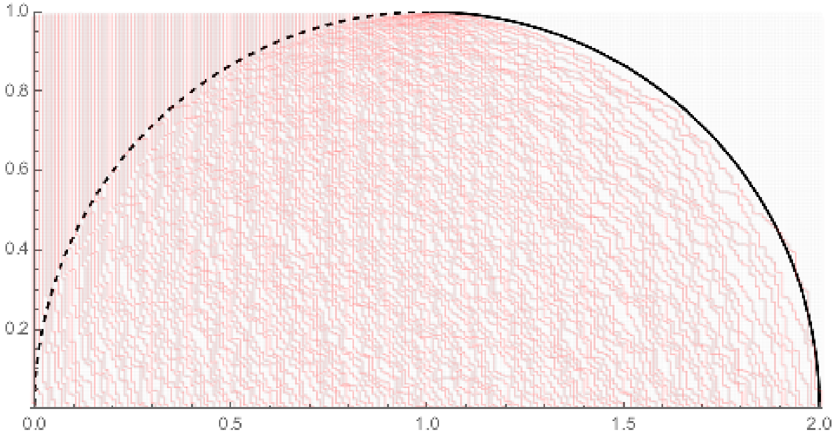
Curve: $Y = \tau\sqrt{2X - X^2}$

$\lambda = (n, n-1, \ldots, 1)$

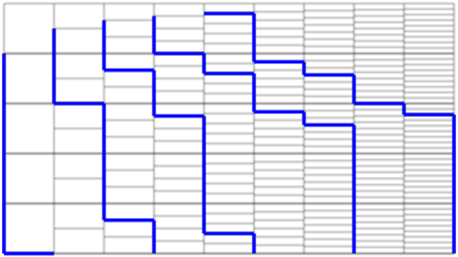


Other Portions of the Arctic Curve

Dual paths: Read tableaux by column rather than by row.

Other Portions of the Arctic Curve

24	23	25	24	25
16	18	17	18	
11	11	2		
6	2			
0				



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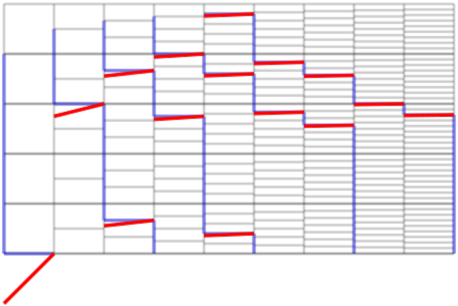

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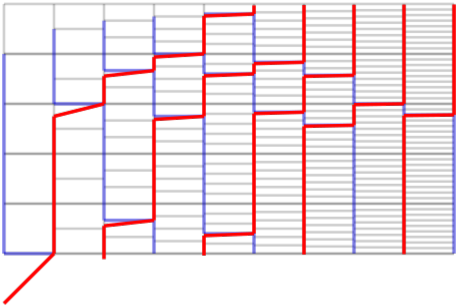

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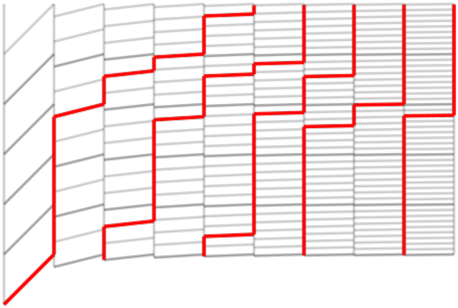

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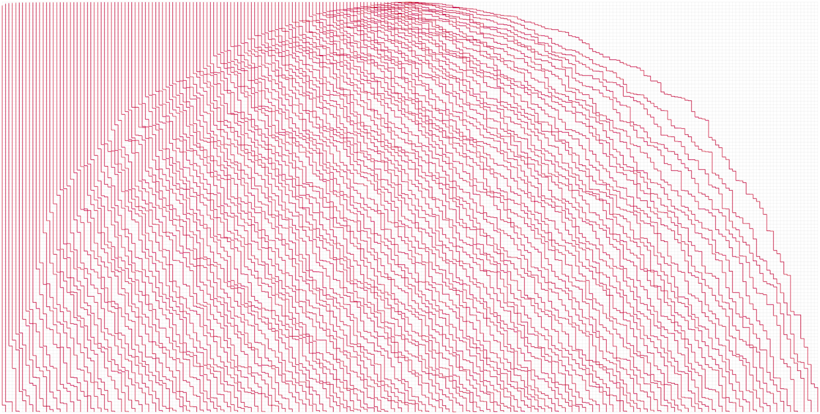
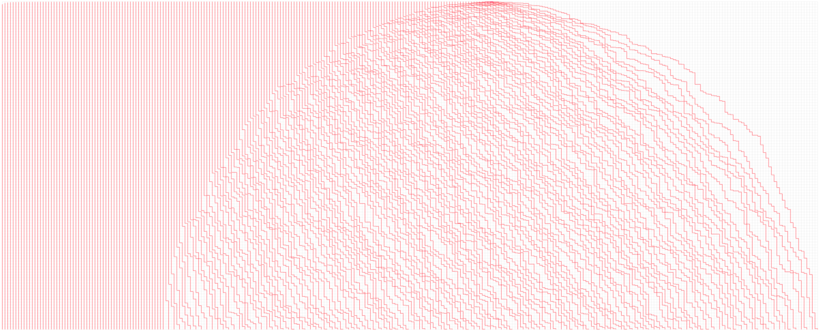


$n = t = 120$

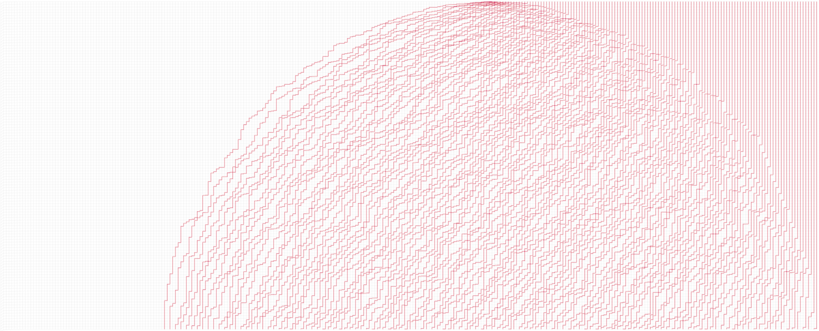
$\lambda = (n, n-1, ..., 1)$

Extend $\lambda = (n, n-1, \ldots, 1, 0, \ldots, 0)$.



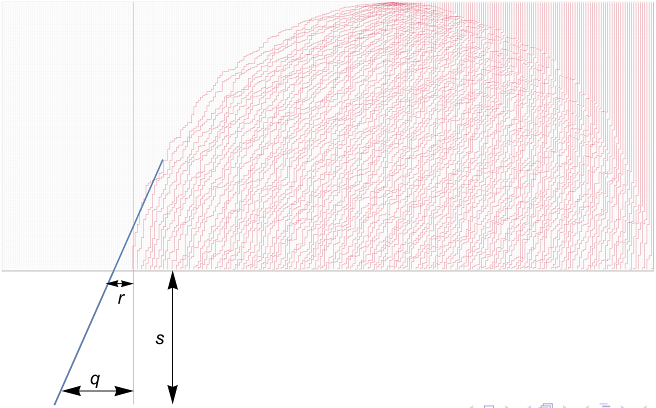
$\lambda = (n, n-1, ..., 1)$

Extend $\lambda = (n, n-1, \ldots, 1, 0, \ldots, 0)$. Switch to dual paths.



$\lambda = (n, n-1, \ldots, 1)$

Fix z, s.
Shifting dual path by r corresponds to
$\lambda = (\lambda_1, \ldots, \lambda_n, 1, \ldots, 1, 0, \ldots, 0)$.



$\lambda = (n, n-1, \ldots, 1)$

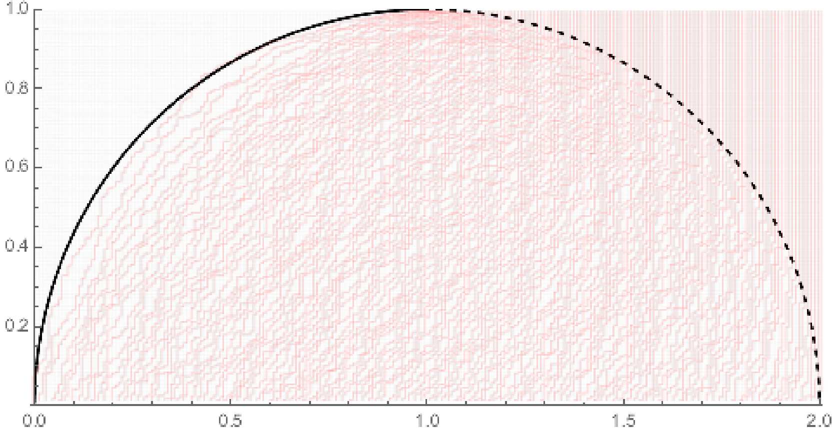
Parametrization:

$$X(x) = \frac{x}{x-1}$$

$$Y(x) = \tau \frac{x}{x-1}\sqrt{\frac{x-2}{x}}$$

for $x \in (-\infty, 0]$.
Same parametrization as before!

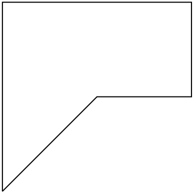
$\lambda = (n, n-1, \ldots, 1)$

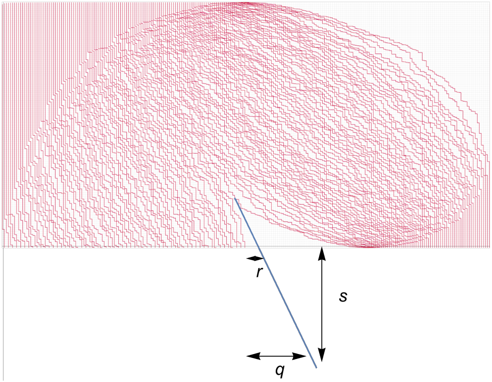


Cusps

Occurs when λ has a macroscopic jump. For example

$$\lambda = (2n, \ldots, 2n, n, n-1, \ldots, 1), t = n$$

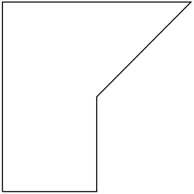


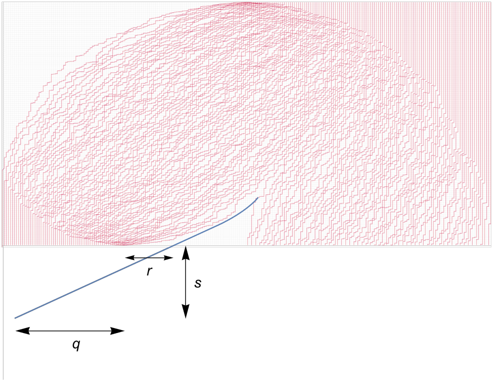


Cusps

Occurs when λ has a macroscopic flat section. For example

$$\lambda = (2n, 2n-1, \dots, n+1, n, \dots, n), t = n$$





In general

For $\lambda = (\lambda_1, \ldots \lambda_n)$ such that $n + \lambda_i - i = n\alpha(\frac{i}{n})$ for some piecewise differentiable α, we have

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Result

The arctic curve can be parametrized by

$$X(x) = \frac{x^2 I'(x)}{I(x) + xI'(x)}$$

$$Y(x) = \tau \frac{1}{I(x) + xI'(x)}$$

for an appropriate range of x*, where* $I(x) = e^{-\int_0^1 \frac{1}{x-\alpha(u)} du}$ *and* $\alpha(u)$ *is the limiting profile.*

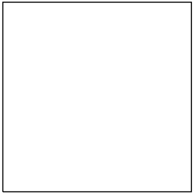
Some Examples

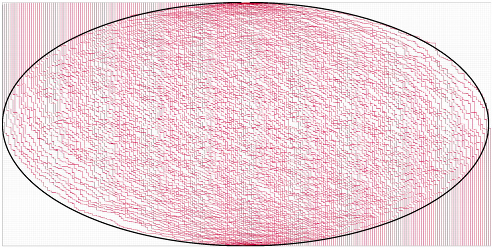
$\lambda = (n, \ldots, n)$

Parametrization:

$$X(x) = \frac{x^2}{x^2 - 2x + 2}$$

$$Y(x) = \frac{\tau(x-1)^2}{x^2 - 2x + 2}$$



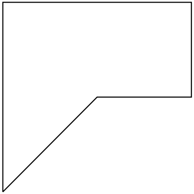


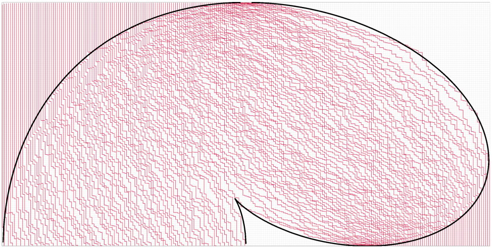
$\lambda = (2n, \ldots, 2n, n, n-1, \ldots, 1)$

Parametrization:

$$X(x) = \frac{x(2x^2 - 9x + 12)}{x^3 - 7x^2 + 17x - 12}$$

$$Y(x) = \frac{\tau(x-3)^2\sqrt{x(x-2)}}{x^3 - 7x^2 + 17x - 12}$$

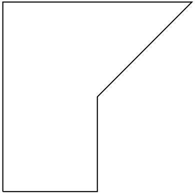


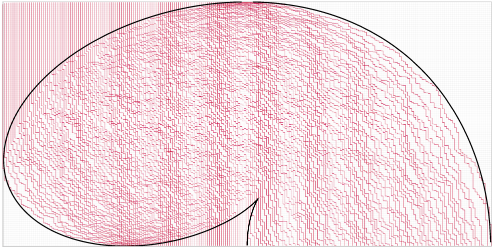


$\lambda = (2n, \ldots, n+1, n, \ldots, n)$

Parametrization:

$$X(x) = \frac{x^2(2x-5)}{x^3 - 5x^2 + 9x - 8}$$
$$Y(x) = \frac{\tau(x-1)^2\sqrt{(x-4)(x-2)}}{x^3 - 5x^2 + 9x - 8}$$



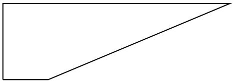


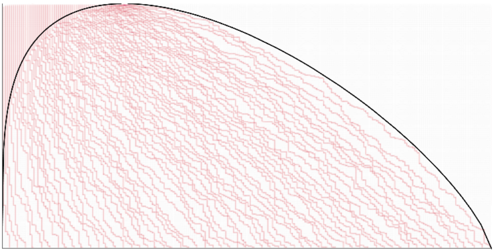
$\lambda = ((p-1)n, (p-1)(n-1), \ldots, (p-1)2, p-1)$

Parametrization:

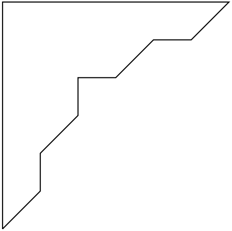
$$X(x) = \frac{x}{x-p+1}$$

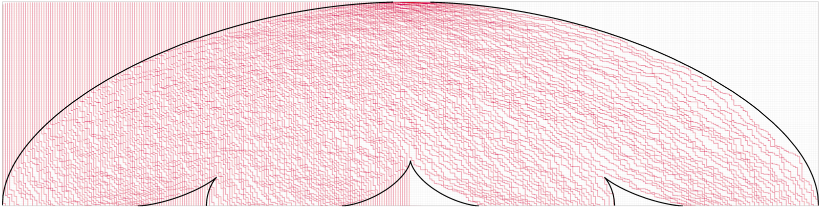
$$Y(x) = \tau \frac{x-p}{x-p+1} \left(\frac{x}{x-p} \right)^{\frac{1}{p}}$$





$\lambda = (6n, \ldots, 5n+1, 4n, \ldots, 3n+1, 2n, \ldots, 2n, 2n, \ldots, n+1, n, \ldots, n, n, \ldots, 1)$





Further Questions

- Skew-tableaux
- q-weighted LHT
- Full limit shape

End!

Thank You!